CS 174: Combinatorics and Discrete Probability (Spring 2023), UC Berkeley

Section 13

1. **(Martingales) (MU Exercise 13.9)** Consider an n-cube with $N = 2^n$ nodes. Let S be a nonempty set of vertices on the cube, and let x be a vertex chosen uniformly at random among all vertices of the cube. Let $D(x, S)$ be the minimum number of coordinates in which x and y differ over all points $y \in S$. Give a bound on:

$$P(|D(x, S) - \mathbb{E}[D(x, S)]| > \lambda) \tag{1}$$

2. **(Martingales) (MU Exercise 13.17)** Given a bag with r red balls and g green balls, suppose that we uniformly sample n balls from the bin without replacement. Set up an appropriate martingale and use it to show that the number of red balls in the sample is tightly concentrated around $nr/(r+g)$.

3. **(Martingales) (MU Exercise 13.19)** Consider a random graph from $G_{n,N}$, where $N = cn$ for some constant $c > 0$. Let X be the number of isolated vertices (vertices of degree 0).

   (a) Determine $\mathbf{E}[X]$.

   (b) Show that

$$P(|X - \mathbf{E}[X]| \geq 2\lambda\sqrt{cn}) \leq 2\exp(-\lambda^2/2) \tag{2}$$

   (Hint: use a martingale that reveals the locations of the edges in the graph, one at a time.)