Section 12

1. **(Martingales) (MU Exercise 13.3** Let $X_0 = 0$ and for $j \geq 0$ let X_{j+1} be chosen uniformly over the real interval $[X_j, 1]$. Show that, for $k \geq 0$, the sequence

$$Y_k = 2^k(1 - X_k) \tag{1}$$

is a martingale.

2. **(Metropolis-Hastings) (MU Exercise 11.12)** The following generalization of the Metropolis algorithm is due to Hastings. Suppose that we have a Markov chain on a state space Ω given by the transition matrix $\mathbf{Q}$ and that we want to construct a Markov chain on this state space with a stationary distribution $\pi_x = b(x)/B$, where for all $x \in \Omega, b(x) > 0$ and $B = \sum_{x\in\Omega} b(x)$ is finite. Define a new Markov chain as follows. When $X_n = x$, generate a random variable Y with $\Pr(Y = y) = Q_{x,y}$. Notice that Y can be generated by simulating one step of the original Markov chain. Set X_{n+1} to Y with probability

$$\min(\frac{\pi_y Q_{y,x}}{\pi_x Q_{x,y}}, 1) \tag{2}$$

and otherwise set X_{n+1} to X_n. Argue that, if this chain is aperiodic and irreducible, then it is also time reversible and has a stationary distribution given by the π_x.

3. **(Martingales) (MU Theorem 13.11)** A parking-lot attendant has mixed up n keys for n cars. The n car owners arrive together. The attendant gives each owner a key according to a permutation chosen uniformly at random from all permutations. If an owner receives the key to his car, he takes it an leaves; otherwise, he return the key to the attendant. The attendant now repeats the process with the remaining keys and car owners. This continues until all owners receive the keys to their cars.

   Let R be the number of rounds until all car owners receive the keys to their cars. We want to compute $\mathbf{E}[R]$. Let X_i be the number of owners who receive their car keys in the ith round. Prove that

$$Y_i = \sum_{j=1}^{i}(X_j - \mathbf{E}[X_j|X_1, ..., X_{j-1}]) \tag{3}$$

   is a martingale. Use the martingale stopping theorem to compute $\mathbf{E}[R]$.