Section 10

1. **(Markov Chains) (MU Exercise 7.3)** Consider a process $X_0, X_1, X_2, ...$ with two states, 0 and 1. The process is governed by two matrices, $\mathbf{P}$ and $\mathbf{Q}$. If k is even, the values $P_{i,j}$ give the probability of going from state i to state j on the step from X_k to X_{k+1}. Likewise, if k is odd then the values $Q_{i,j}$ give the probability of going from state i to state j on the step from X_k to X_{k+1}. Explain why this process does not satisfy Definition 7.1 of a (time-homogeneous) Markov chain. Then give a process with a larger state space that is equivalent to this process and satisfies Definition 7.1.

2. **(Markov Chains) (MU Exercise 7.13)** Consider a finite Markov chain on n states with stationary distribution π and transition probabilities $P_{i,j}$. Imagine starting the chain at time 0 and running it for m steps, obtaining the sequence of states $X_0, X_1, ..., X_m$. Consider the states in reverse order, $X_m, X_{m-1}, ..., X_0$.

   (a) Argue that given X_{k+1}, the state X_k is independent of $X_{k+2}, X_{k+3}, ..., X_m$. Thus the reverse sequence is Markovian.

   (b) Argue that for the reverse sequence, the transition probabilities $Q_{i,j}$ are given by

   $$Q_{i,j} = \frac{\pi_i P_{j,i}}{\pi_j} \tag{1}$$

   (c) Prove that if the original Markov chain is time reversible, so that $\pi_i P_{i,j} = \pi_j P_{j,i}$, then $Q_{i,j} = P_{i,j}$. That is, the states follow the same transition probabilities whether viewed in forward order or reverse order.

3. **(Markov Chains) (MU Exercise 7.22)** A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of $O(m^2 n)$ on the expected time before the cat eats the mouse. (Hint: Consider a Markov chain whose states are the ordered pairs (a, b), where a is the position of the cat and b is the position of the mouse.)