Consider a Markov chain on $n$ points $[0, n-1]$ lying in order on a circle, equally spaced out. At each step, the chain stays at the current position with probability $1/2$, or every point moves to the next point in the clockwise direction with probability $1/2$.

(a) Show that the chain is irreducible and aperiodic.

(b) What is the stationary distribution? Justify your answer.

Here is a coupling $(X_t, Y_t)$ for this process. At each step, if $X_t \neq Y_t$, then with probability $1/2$, $X_t$ stays at the current position and every point on $Y_t$ moves clockwise, and with probability $1/2$, vice versa. If $X_t = Y_t$, then with probability $1/2$, both chains stay at the current position, and with probability $1/2$, both chains move clockwise. This is clearly a valid coupling.

(c) Show that for any choice of initial states $X_0, Y_0$, the expected number of steps $T$ until $X_T = Y_T$ is $O(n^2)$. [HINT: Compare with random walk on a line.]

(d) Deduce that the mixing time satisfies $\tau(\epsilon)$ is $O(n^2/\epsilon)$ [HINT: Use Markov’s inequality and the coupling lemma.]