Homework 9

Out: 4 Apr. Due: 11 Apr.

Submit your solutions in pdf format on Gradescope by **5pm on Friday, April 11**. Solutions may be written either in $\&T_EX$ (with either machine-drawn or hand-drawn diagrams) or **legibly** by hand. (The $\&T_EX$ source for this homework is provided in case you want to use it as a template.) Please be sure to begin the solution for each problem on a new page, and to tag each of your solutions to the correct problem! Per course policy, no late solutions will be accepted. Take time to write **clear** and **concise** answers; confused and long-winded solutions may be penalized. You are encouraged to form small groups (two to four people) to work through the homework, but you **must** write up all your solutions on your own. Depending on grading resources, we reserve the right to grade a random subset of the problems and check off the rest; so you are advised to attempt all the problems.

1. [Variant of MU, Exercise 7.1] Consider the Markov chain with four states $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 3/10 & 1/10 & 3/5\\ 1/10 & 1/10 & 7/10 & 1/10\\ 1/10 & 7/10 & 1/10 & 1/10\\ 9/10 & 1/10 & 0 & 0 \end{pmatrix}$$

Thus $P_{1,4} = 3/5$ is the probability of moving from state 1 to state 4.

- (a) Find the probability of being in state 4 after 3 steps if the chain begins in state 1. [HINT: Do this by hand; you can do it *without* multiplying matrices!]
- (b) Find the probability of being in state 4 after 3 steps if the chain begins at a state chosen u.a.r. from all four states. [HINT: Again, do this by hand.]
- (c) Find the stationary distribution π of this chain. [NOTE: You will probably need to use a linear algebra package for this.]
- (d) Suppose the chain begins in state 1. What is the smallest value of t for which the variation distance $||p_1^t \pi||$ is less than 0.001? [NOTE: Recall that p_x^t denotes the distribution of the chain after t steps starting from state x. Again, use the package.]
- 2. In this problem we will classify the states of the 1-dimensional random walk, i.e., the Markov chain whose state space consists of the set of all integers \mathbb{Z} , and whose transition probabilities from any state *i* are $P_{i,i+1} = p$, $P_{i,i-1} = q = 1 p$. (Note that this chain is periodic with period 2.)
 - (a) For an arbitrary Markov chain, prove the following characterization of transient/recurrent states: state *i* is transient iff¹ $\sum_{t} P_{i,i}^{t} < \infty$ (equivalently, state *i* is recurrent iff $\sum_{t} P_{i,i}^{t} = \infty$). [HINT: Let the r.v. T_{i} denote the return time from *i* back to itself, and define $f_{i} := \Pr[T_{i} < \infty]$. Also let V_{i} denote the total number of visits to *i* when starting from *i*. Show that $E[V_{i}] = \infty$ iff $f_{i} = 1$. To do this, recall that $E[X] = \sum_{k\geq 1} \Pr[X \geq k]$ for any nonnegative integer-valued r.v. *X*. What is the relationship between $E[V_{i}]$ and $\sum_{t} P_{i,i}^{t}$?]
 - (b) For the random walk above, started in state 0, show that the probability that the walk is at 0 after 2t steps is $P_{0,0}^{2t} \sim \frac{(4pq)^t}{\sqrt{\pi t}}$. [HINT: Use Stirling's formula to show that $\binom{2t}{t} \sim \frac{2^{2t}}{\sqrt{\pi t}}$.]
 - (c) Conclude from parts (a) and (b) that the random walk is recurrent when $p = q = \frac{1}{2}$ and transient otherwise.
 - (d) Prove that, when $p = q = \frac{1}{2}$, the chain is in fact null recurrent. [HINT: For i > 1, let H_i denote the expected hitting time from state i to 0. Write down a system of equations for the H_i , and show that the solution to these equations (if finite) satisfies $H_i = iH_1 i(i-1)$ for $i \ge 1$.]

[continued on next page]

¹This is not a typo! The term "iff" stands for "if and only if."

3. This question concerns the "lollipop" graph L_n , which consists of a clique on $\frac{n}{2}$ vertices with a "tail" of length $\frac{n}{2}$ (edges) attached (so the total number of vertices is n). The tail is attached to the clique at vertex a, and the end of the tail is vertex b (see Figure). We assume that n is even and $n \ge 6$.



We also use the following notation for random walk on any undirected graph G = (V, E):

- For any two vertices $u, v \in V$, $H_{u,v}$ denotes the expected hitting time from u to v (i.e., the expected number of steps until the walk, starting at u, reaches v).
- For any vertex $v \in V$, $C_v(G)$ denotes the cover time from v (i.e., the expected time for the walk, starting at v, to visit all vertices of G).
- $C(G) = \max_v C_v(G)$ denotes the cover time of G.

In the following questions, you may assume without proof any results we have derived in class provided you state them clearly. Also, remember that a $\Theta(\cdot)$ expression is *both* an upper *and* a lower bound (up to constant factors and lower order terms).

- (a) Let K_n be the complete graph on *n* vertices. Show that $C_v(K_n) = \Theta(n \log n)$ for all vertices *v* of K_n .
- (b) For the lollipop graph L_n , show that $C(L_n) = O(n^3)$. [NOTE: You are only asked to show an *upper* bound in this part.]
- (c) Show that $C_b(L_n) = \Theta(n^2)$. [NOTE: This is both an upper and a lower bound.]
- (d) For the lollipop graph L_n , show that $H_{a,b}$ satisfies

$$H_{a,b} \ge \frac{1}{n/2} \Big(1 - \frac{2}{n} \Big) H_{a,b} + \frac{n/2 - 1}{n/2} \Big(H_{a,b} + \Omega(n) \Big).$$

[HINT: What does Gambler's Ruin say about the probability that random walk on the line $\{0, \ldots, n/2\}$, starting from 1, hits 0 before hitting n/2?] Deduce that $H_{a,b} = \Omega(n^3)$.

- (e) Deduce from parts (b) and (d) that $C(L_n) = \Theta(n^3)$. [NOTE: Again, both an upper and a lower bound.]
- (f) Prove or disprove the following statement: "If G is a connected graph and G' is obtained from G by adding edges to G, then $C(G) \leq C(G')$."
- (g) Prove or disprove the following statement: "If G is a connected graph and G' is obtained from G by adding edges to G, then $C(G) \ge C(G')$."
- 4. The exclusion process on the directed cycle is a Markov chain defined as follows. There are n sites corresponding to the vertices of the cycle C_n , and 1 < k < n indistinguishable particles which may occupy the sites, with at most one particle per site. Thus there are $\binom{n}{k}$ allowed configurations of particles. Transitions from any configuration are specified as follows:
 - pick a particle u.a.r.
 - move the particle one position clockwise round the cycle, provided that site is not occupied; else do nothing
 - (a) Explain briefly why the process is irreducible.
 - (b) Explain briefly why the process is aperiodic.
 - (c) What is the stationary distribution? Justify your answer.