Homework 9

Out: 14 Apr. Due: 21 Apr.

Submit your solutions in pdf format on Gradescope by **5pm on Friday, April 21**. *Solutions may be written either in LaTeX (with either machine-drawn or hand-drawn diagrams) or* **legibly** *by hand. (The LaTeX source for this homework is provided in case you want to use it as a template.) Please be sure to begin the solution for each problem on a new page, and to tag each of your solutions to the correct problem! Per course policy, no late solutions will be accepted. Take time to write* **clear** *and* **concise** *answers; confused and long-winded solutions may be penalized. You are encouraged to form small groups (two to four people) to work through the homework, but you* **must** *write up all your solutions on your own. Depending on grading resources, we reserve the right to grade a random subset of the problems and check off the rest; so you are advised to attempt all the problems.*

1. Let $G = (V, E)$ be a connected graph, and let $\mathcal{T}$ denote the set of all *spanning trees* of G. (Note that $|\mathcal{T}|$ is in general exponentially large as a function of the size of G.) Consider the Markov chain on state space $\mathcal{T}$ which makes transitions from its current state (a tree T) as follows:

- pick an edge $e \in E \setminus T$ u.a.r.
- add e to T, creating a unique cycle C
- remove one edge of C u.a.r., obtaining a new tree T'

(a) Show that the above Markov chain is ergodic (i.e., irreducible and aperiodic) and that its stationary distribution is uniform over $\mathcal{T}$.

(b) Explain how to modify the transition probabilities of the chain so that its stationary distribution is $\pi(T) \propto \prod_{e \in T} \lambda_e$, where $\{\lambda_e : e \in E\}$ are positive edge weights. Be sure to justify why this new distribution is indeed stationary.

2. Recall from class and MU Definition 11.2 the definition of a *fully polynomial randomized approximation scheme (fpras)* for a non-negative function $f(x)$. This is a randomized algorithm that takes as input a triple (x, ε, δ), outputs a value in the range $[(1-\varepsilon)f(x), (1+\varepsilon)f(x)]$ with probability at least $1 - \delta$, and runs in time polynomial in $|x|$, $1/\varepsilon$ and $\log \delta^{-1}$.

Show that we may simplify this definition as follows: the algorithm takes as input a pair (x, ε), outputs a value in the above range with probability at least $3/4$, and runs in time polynomial in $|x|$ and $1/\varepsilon$. In other words, what you have to show here is that any algorithm of this simpler form can be modified into an fpras. Be sure to prove that your modified algorithm has the required properties. [HINT: Consider performing repeated trials and taking the median; use a Chernoff bound. Note that this problem has nothing to do with the randomized median-finding algorithm discussed in Lecture 6!]

[continued on next page]

3. A sequence (Z_t) of random variables is called a *submartingale* w.r.t. some other sequence (X_t) if it satisfies the same conditions as a martingale with the conditional expectation statement replaced by

$$\mathrm{E}[Z_{t+1} \mid X_0, \ldots, X_t] \geq Z_t.$$

The Optional Stopping Theorem holds for submartingales, with the conclusion $\mathrm{E}[Z_T] \geq Z_0$ for a suitable stopping time T. [It also holds for supermartingales, with all the inequalities reversed.]

Now consider a process (X_t) on the integers that starts at $X_0 = 0$ and at each step t increments/decrements its position according to an integer-valued r.v. D_t such that

$$\mathrm{E}[D_t \mid X_{t-1}] = 0; \qquad \mathrm{E}[D_t^2 \mid X_{t-1}] \geq \alpha; \qquad |D_t| \leq c \tag{$*$}$$

for all t, where α is a positive real and c is a positive integer. Thus the position of the process at time t is given by $X_t = X_0 + \sum_{i=1}^{t} D_t$. Clearly (X_t) is a martingale w.r.t. (D_t) by the first property in $(*)$. Answer the following questions about this process.

(a) Let T be the time until the process first exits the interval $[-m, m]$ (where m is a positive integer), and let p denote the probability that this exit occurs at the right-hand end (i.e., $X_T > m$). Use the Optional Stopping Theorem to show that $\frac{m+1}{2m+c+1} \leq p \leq \frac{m+c}{2m+c+1}$.

(b) Define $Z_t = X_t^2 - \alpha t$. Show that (Z_t) is a submartingale w.r.t. (D_t).

(c) Use the Optional Stopping Theorem for submartingales to show that $\mathrm{E}[T] \leq \frac{(m+c)^2}{\alpha}$.

4. Let $G = (V, E)$ be an arbitrary connected undirected graph with n vertices and m edges. Consider the following random process on G, which is a very simple model for the spread of rumors or infections. Initially, each vertex of G is colored either black or white (with no constraints). Then, at each step, all vertices simultaneously update their colors, independently of all other vertices, as follows:

- with probability $\frac{1}{2}$ do nothing
- else pick a neighboring vertex u.a.r. and adopt the color of that vertex

(Note that *all* vertices make these decisions before any vertex changes color.) It should be clear that this process will eventually terminate with all vertices black or all white, after which no further change is possible.

(a) Let the random variable X_t denote the sum of the degrees of all the white vertices at time t. Show that (X_t) is a martingale with respect to the sequence (Y_t), where Y_t denotes the outcomes of the tth step of the process.

(b) Use the Optional Stopping Theorem to show that the probability that the process terminates in the all-white configuration is $\frac{X_0}{2m}$, where X_0 is the sum of degrees of white vertices at time 0. [NOTE: Don't forget to justify why the OST holds!]

(c) Use the Optional Stopping Theorem again to show that the expected duration of the process, starting from any initial configuration, is at most $O(m^2)$ steps. [NOTE: Part (c) of the previous question should be useful here. Justify any claims you make about the conditional variance of X_t.]