Homework 8

Out: 7 Apr. Due: 14 Apr.

Submit your solutions in pdf format on Gradescope by **5pm on Friday, April 14**. *Solutions may be written either in LaTeX (with either machine-drawn or hand-drawn diagrams) or* **legibly** *by hand. (The LaTeX source for this homework is provided in case you want to use it as a template.) Please be sure to begin the solution for each problem on a new page, and to tag each of your solutions to the correct problem! Per course policy, no late solutions will be accepted. Take time to write* **clear** *and* **concise** *answers; confused and long-winded solutions may be penalized. You are encouraged to form small groups (two to four people) to work through the homework, but you* **must** *write up all your solutions on your own. Depending on grading resources, we reserve the right to grade a random subset of the problems and check off the rest; so you are advised to attempt all the problems.*

1. [Variant of MU, Exercise 7.1] Consider the Markov chain with four states $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 3/10 & 1/10 & 3/5 \\ 1/10 & 1/10 & 7/10 & 1/10 \\ 1/10 & 7/10 & 1/10 & 1/10 \\ 9/10 & 1/10 & 0 & 0 \end{pmatrix}$$

Thus $P_{1,4} = 3/5$ is the probability of moving from state 1 to state 4.

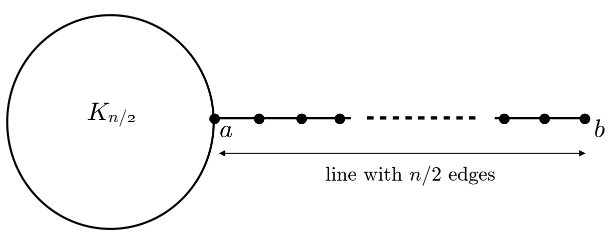
(a) Find the probability of being in state 4 after 3 steps if the chain begins in state 1. [HINT: Do this by hand; you can do it *without* multiplying matrices!]

(b) Find the probability of being in state 4 after 3 steps if the chain begins at a state chosen u.a.r. from all four states. [HINT: Again, do this by hand.]

(c) Find the stationary distribution π of this chain. [NOTE: You will probably need to use a linear algebra package for this.]

(d) Suppose the chain begins in state 1. What is the smallest value of t for which the variation distance $\|p_1^t - \pi\|$ is less than 0.001? [NOTE: Recall that p_x^t denotes the distribution of the chain after t steps starting from state x. Again, use the package.]

2. This question concerns the "lollipop" graph L_n, which consists of a clique on $\frac{n}{2}$ vertices with a "tail" of length $\frac{n}{2}$ (edges) attached (so the total number of vertices is n). The tail is attached to the clique at vertex a, and the end of the tail is vertex b (see Figure). We assume that n is even and $n \geq 6$.



We also use the following notation for random walk on any undirected graph $G = (V, E)$:

- For any two vertices $u, v \in V$, H_{uv} denotes the expected hitting time from u to v (i.e., the expected number of steps until the walk, starting at u, reaches v).
- For any vertex $v \in V$, $C_v(G)$ denotes the cover time from v (i.e., the expected time for the walk, starting at v, to visit all vertices of G).
- $C(G) = \max_v C_v(G)$ denotes the cover time of G.

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In the following questions, you may assume without proof any results we have derived in class provided you state them clearly. Also, remember that a $\Theta(\cdot)$ expression is *both* an upper *and* a lower bound.

(a) Let K_n be the complete graph on n vertices. Show that $C_v(K_n) = \Theta(n \log n)$ for all vertices v of K_n.

(b) For the lollipop graph L_n, show that $C(L_n) = O(n^3)$. [NOTE: You are only asked to show an *upper* bound in this part.]

(c) Show that $C_b(L_n) = \Theta(n^2)$. [NOTE: This is both an upper and a lower bound.]

(d) For the lollipop graph L_n, show that $H_{a,b}$ satisfies

$$H_{a,b} \geq \frac{1}{n/2}\Big(1 - \frac{2}{n}\Big)H_{a,b} + \frac{n/2-1}{n/2}\Big(H_{a,b} + \Omega(n)\Big).$$

[HINT: What does Gambler's Ruin say about the probability that random walk on the line $\{0, \ldots, n/2\}$, starting from 1, hits 0 before hitting $n/2$?] Deduce that $H_{a,b} = \Omega(n^3)$.

(e) Deduce from parts (b) and (d) that $C(L_n) = \Theta(n^3)$. [NOTE: Again, both an upper and a lower bound.]

(f) Prove or disprove the following statement: "If G is a connected graph and G' is obtained from G by adding edges to G, then $C(G) \leq C(G')$."

(g) Prove or disprove the following statement: "If G is a connected graph and G' is obtained from G by adding edges to G, then $C(G) \geq C(G')$."

3. The *exclusion process on the directed cycle* is a Markov chain defined as follows. There are n sites corresponding to the vertices of the cycle C_n, and $1 < k < n$ indistinguishable particles which may occupy the sites, with at most one particle per site. Thus there are $\binom{n}{k}$ allowed configurations of particles. Transitions from any configuration are specified as follows:

- pick a particle u.a.r.
- move the particle one position clockwise round the cycle, provided that site is not occupied; else do nothing

(a) Explain briefly why the process is irreducible.

(b) Explain briefly why the process is aperiodic.

(c) What is the stationary distribution? Justify your answer.

4. Recall the "random transpositions" card shuffle that we defined in class. Here the states, as usual, are all $n!$ permutations of an n-card deck, and at each step the shuffle proceeds as follows:

- pick two positions, $i, j \in \{1, \ldots, n\}$ independently and u.a.r. (note that $i = j$ is possible)
- swap the cards at positions i and j

As we saw in class, this shuffle converges to the uniform distribution. (It is irreducible because any permutation can be written as the product of transpositions; it is aperiodic because there is a self-loop probability of $1/n$ at each state; and the stationary distribution is uniform because the transition probabilities are symmetric.)

In this problem you will show that $O(n^2)$ shuffles are enough to mix up the deck.

Here is a coupling (X_t, Y_t) for this process. At each step, we choose a position $i \in \{1, \ldots, n\}$ and a card c u.a.r. Then in both copies X_t, Y_t we swap card c with the card in position i. (Note that this is a valid coupling, because both copies, viewed separately, are in fact swapping the cards in two randomly chosen positions, as specified in the original process.)

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To analyze this coupling, let $d_t = d(X_t, Y_t)$ be the *distance* between the two copies after t steps, i.e., the number of cards whose positions differ in X_t and Y_t.

(a) Explain *carefully* why d_t never increases with t.

(b) Show that d_t decreases by at least 1 with probability $\left(\frac{d_t}{n}\right)^2$.

(c) Deduce that, for any choice of initial states X_0, Y_0, the expected number of steps T until $X_T = Y_T$ is at most cn^2 for some constant c. [HINT: Recall the expected value of a geometric r.v. Recall also that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$.]

(d) Finally, deduce that the mixing time satisfies $\tau(\varepsilon) \leq \frac{cn^2}{\varepsilon}$. [HINT: Use Markov's inequality and the coupling lemma. In fact, the mixing time for this process satisfies $\tau(\varepsilon) \leq cn^2 \log(\frac{1}{\varepsilon})$, but you are not required to prove this.]

5. [Optional extra: No credit] Here is an unusual card trick. I take a shuffled deck and turn up the cards one by one. I ask you to select one of the first ten cards, without telling me which one; let $c_1 \in \{1, 2, \ldots, 13\}$ be the numerical value of your card. You then count c_1 cards from the one you selected, and note that card; call its value c_2. You then count a further c_2 cards and note that card, and so on until the deck is exhausted. At that point, I am able to identify the last card you noted (at least most of the time).

Describe how I perform this amazing feat, and give a qualitative explanation for why it works. [HINT: think about coupling. You are not expected to perform any calculations to justify why the method works. You are encouraged to try it on a friend a few times and estimate the success probability—it should certainly be enough to win comfortably in a gambling situation. Or if you are really interested you could simulate the trick with a program and get a much better estimate of the success probability.]