Homework 1

Out: 20 Jan. Due: 27 Jan.

Submit your solutions in pdf format on Gradescope by **5pm on Friday, January 27**. *Solutions may be written either in LaTeX (with either machine-drawn or hand-drawn diagrams) or* **legibly** *by hand. (The LaTeX source for this homework is provided in case you want to use it as a template.) Please be sure to begin the solution for each problem on a new page, and to tag each of your solutions to the correct problem! Per course policy, no late solutions will be accepted. Take time to write* **clear** *and* **concise** *answers; confused and long-winded solutions may be penalized. You are encouraged to form small groups (two to four people) to work through the homework, but you* **must** *write up all your solutions on your own. Depending on grading resources, we reserve the right to grade a random subset of the problems and check off the rest; so you are advised to attempt all the problems.*

1. You are dealt a hand of five cards from a randomly shuffled deck. Compute the probabilities of each of the following events, explaining your reasoning in each case.

(a) Your hand contains at least one ace.

(b) The highest card in your hand is 10. (Assume that ace is the highest card.)

(c) Your hand is a flush (i.e., all cards are from the same suit).

(d) Your hand is a full house (i.e., it contains three cards of one value and two of some other value).

2. Suppose we roll ten unbiased 6-sided dice. What is the probability that the sum of the pips on the dice is divisible by 3? [HINT: Use the principle of deferred decisions. Justify your answer carefully: avoid hand-waving!]

3. A fair coin is tossed $2n$ times. What is the probability that we observe a consecutive sequence of at least n heads?

4. You are given an urn containing 10 balls, some black and some white. You are told that the number of white balls in the urn is equally likely to be any number between 0 and 10 inclusive. Thus in particular the probability that all 10 balls are white is $\frac{1}{11}$.

(a) You pick a ball uniformly at random and it is white. Given this event, what is now the probability that all 10 balls are white? [HINT: You will need to use Bayes' rule: $\Pr[A|B] = \frac{\Pr[B|A]\Pr[A]}{\Pr[B]}$.]

(b) You pick k balls at random (with replacement) and all are white. What is now the probability that all 10 balls are white, as a function of k?

5. Here are some problems based on the randomized min-cut algorithm discussed in class (MU Section 1.5).

(a) A graph may have more than one minimum cut. Using the analysis of the error probability of the randomized min-cut algorithm, show that the number of distinct minimum cuts is at most $\frac{n(n-1)}{2}$.

(b) Suppose that the algorithm is modified as follows. Rather than picking an *edge* uniformly at random and merging its endpoints, the algorithm picks a pair of vertices (not necessarily adjacent) u.a.r. and merges them. Give a family of connected graphs G_n (where G_n has n vertices for each n) such that when the modified algorithm is run on G_n the probability that it finds a minimum cut is *exponentially* small in n. [NOTE: By "exponentially small" we mean that the probability is less than c^{-n} for some constant $c > 1$ and all sufficiently large n.]

(c) Show that an exponential number of repeated trials of the algorithm of part (b) would be needed in order to reduce the error probability to $\frac{1}{2}$.