Homework 8

Out: 19 Mar. Due: 2 Apr.

Instructions: *Submit your solutions in pdf format on Gradescope by* **5pm on Friday, April 2**. *Solutions may be written either in LaTeX (with either machine-drawn or hand-drawn diagrams) or* **legibly** *by hand. (The LaTeX source for this homework is provided in case you want to use it as a template.) Please be sure to begin the solution for each problem on a new page, and to tag each of your solutions to the correct problem! Per course policy, no late solutions will be accepted. Take time to write* **clear** *and* **concise** *answers; confused and long-winded solutions may be penalized. You are encouraged to form small groups (two to four people) to work through the homework, but you* **must** *write up all your solutions on your own. Depending on grading resources, we reserve the right to grade a random subset of the problems and check off the rest; so you are advised to attempt all the problems.*

1. (a) The COLORABILITY problem is the following.

INSTANCE: An undirected graph $G = (V, E)$, and a positive integer k.

QUESTION: Is it possible to color the vertices of G with at most k different colors, in such a way that the endpoints of every edge receive different colors?

Consider the following mapping from instances of 3-SAT to instances of COLORABILITY. Let ϕ be an arbitrary formula in 3-CNF, with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_r$. Given ϕ, construct a graph $G = (V, E)$, with

$$\begin{aligned} V &= \{v_0, \ldots, v_n\} \cup \{x_1, \ldots, x_n\} \cup \{\overline{x}_1, \ldots, \overline{x}_n\} \cup \{C_1, \ldots, C_r\}\ ; \\ E &= \big\{\{v_i, x_j\}, \{v_i, \overline{x}_j\} : 1 \le i, j \le n \text{ and } i \ne j\big\} \\ &\quad \cup \big\{\{v_i, v_j\} : 0 \le i < j \le n\big\} \\ &\quad \cup \big\{\{v_0, C_k\} : 1 \le k \le r\big\} \\ &\quad \cup \big\{\{x_i, \overline{x}_i\} : 1 \le i \le n\big\} \\ &\quad \cup \big\{\{x_i, C_k\} : x_i \text{ is } \textit{not} \text{ a literal in clause } C_k\big\} \\ &\quad \cup \big\{\{\overline{x}_i, C_k\} : \overline{x}_i \text{ is } \textit{not} \text{ a literal in clause } C_k\big\}. \end{aligned}$$

The mapping in question takes ϕ and outputs the graph G and the integer $k = n + 1$.

(i) Draw the graph G obtained from the formula

$$\phi = (\overline{x}_1 \vee\ x_2 \vee x_3) \wedge (x_1 \vee \overline{x}_2 \vee x_3) \wedge (x_1 \vee \overline{x}_2 \vee \overline{x}_3).$$

NOTE: Give some thought to a helpful layout of your graph.

(ii) Show that the above mapping is a polynomial time reduction from SAT to COLORABILITY.

HINT: Your main task is to show that G can be colored with k colors if and only if ϕ is satisfiable. The $n + 1$ 'new' vertices $v_0, v_1, \ldots, v_n$ are intended to play the role of a 'palette' of $n + 1$ colors. For each variable x_i, one of the vertices x_i and $\overline{x}_i$ is intended to receive the same color as v_i (color i) and the other the same color as v_0 (color 0), according as to whether x_i is true or false.

(iii) Deduce that COLORABILITY is NP-complete.

(b) The following problem, TIMETABLE, arises in the timetabling of examination papers.

INSTANCE: A set P of exams, a set S of timetable "slots", a set $C = \{c_1, \ldots, c_k\}$ of candidates, and, for each candidate c_i, a set $P_i \subseteq P$ of exams that the candidate is expecting to take.

QUESTION: Is there an assignment of the exams to timetable slots that avoids clashes? (A clash occurs if some candidate is required to sit two exams simultaneously.)

By presenting a reduction from COLORABILITY, show that TIMETABLE is NP-complete.

NOTE: Here — as in many applications of the theory of NP-completeness to "real life" problems — you will be using a very special case of the target problem TIMETABLE.

2. Show that if $\mathrm{P} = \mathrm{NP}$ then every language in NP except for the trivial languages $\emptyset$ and Σ^* is NP-complete. Why do we need to exclude these trivial languages?

3. The *Integer Programming* problem, IP, is the following:

INSTANCE: A system of linear inequalities in the variables $x_1, x_2, \ldots, x_n$.

QUESTION: Does the system of inequalities have a solution in which all the x_i are integers?

[A linear inequality is an expression of the form $\sum_i a_i x_i \leq b$, where the coefficients a_i and b are integers. You may recall from CS170 the *Linear Programming* problem, which is defined in exactly the same way except that the solution is not required to be integer-valued. Linear Programming is in P.]

(a) By presenting a reduction from VERTEXCOVER, prove that IP is NP-hard[1].

[NOTES: (i) You may find it convenient to introduce inequalities of the form $x \leq 1$ and $-x \leq 0$, which force the variable x to be either 0 or 1.]

(b) To show that IP is NP-complete, we need to show in addition that IP $\in \mathrm{NP}$. For most languages we have discussed, it is more or less obvious that the language belongs to NP. Explain briefly but carefully why this is not so obvious in the case of IP.

[NOTE: In fact it can be shown that IP $\in \mathrm{NP}$. If you have a strong background in linear algebra, you might like to try to prove this. The sample solutions will contain a reference to a proof in the literature.]

[1]We say that a language L is *NP-hard* if every language $L' \in \mathrm{NP}$ is polynomial time reducible to L. Thus L is NP-complete iff L is NP-hard and also $L \in \mathrm{NP}$.