CS172 Computability & Complexity, Spring 2021

Homework 11

Out: 23 Apr. Due: 30 Apr.

Instructions: *Submit your solutions in pdf format on Gradescope by* **5pm on Friday, April 23**. *Solutions may be written either in LaTeX (with either machine-drawn or hand-drawn diagrams) or* **legibly** *by hand. (The LaTeX source for this homework is provided in case you want to use it as a template.) Please be sure to begin the solution for each problem on a new page, and to tag each of your solutions to the correct problem! Per course policy, no late solutions will be accepted. Take time to write* **clear** *and* **concise** *answers; confused and long-winded solutions may be penalized. You are encouraged to form small groups (two to four people) to work through the homework, but you* **must** *write up all your solutions on your own. Depending on grading resources, we reserve the right to grade a random subset of the problems and check off the rest; so you are advised to attempt all the problems.*

1. This problem considers a version of the Space Hierarchy Theorem for *nondeterministic* space: i.e., we would like to prove that, for any space constructible function f, there exists a language that is in $\text{NSPACE}(f(n))$ but not in $\text{NSPACE}(g(n))$ for any $g(n) = o(f(n))$.

(a) First, let's prove a rather weak nondeterministic Space Hierarchy Theorem. Show that, for any function $f(n) \geq (\log n)^2$, there exists a language that is in $\text{NSPACE}(f(n))$ but not in $\text{NSPACE}(g(n))$ for any $g(n) = o(\sqrt{f(n)})$.

[HINT: Use Savitch's Theorem. Recall that Savitch's Theorem actually holds for any space bound $S(n) \geq \log n$.]

(b) Explain precisely why the proof of the Space Hierarchy Theorem (Theorem 9.3 in Sipser) breaks down for nondeterministic space.

[NOTE: You just need to identify one key point here.]

(c) In 1987, Immerman and Szelepcsényi (independently) proved that, for any space-constructible function $f(n) \geq \log n$, the class $\text{NSPACE}(f(n))$ is closed under complementation, i.e., the complement of every language in $\text{NSPACE}(f(n))$ also belongs to $\text{NSPACE}(f(n))$. Assuming this fact[1], show how the proof of the Space Hierarchy Theorem may be adapted to prove the strong hierarchy theorem for nondeterministic space stated at the top of this problem.

[HINT: Instead of making the simulating TM *disagree* with M, as in the proof of Theorem 9.3, make it *agree*! Then use the above closure result to obtain a contradiction. You are **not** required to repeat all the proof details again: it is sufficient to just indicate the main differences.]

[continued on next page]

[1] In case you are interested, a proof for the case $f(n) = \log n$ can be found in Section 8.6 of Sipser; the proof for general f is similar. You don't need to know anything about this proof in order to answer this question. Note that it is easy to deduce from Savitch's Theorem the weaker fact that the complement of any language in $\text{NSPACE}(f(n))$ belongs to $\text{NSPACE}(f(n)^2)$.

2. (a) Show that if $\textsc{Space}(n) \subseteq \mathrm{P}$ then $\mathrm{P} = \mathrm{P}\textsc{Space}$.

[Hint: For any language $L \in \textsc{Space}(n^k)$ for some natural number k, consider the language $L_{\text{pad}} = \{x\$^{|x|^k} : x \in L\}$. Show that L_{pad} belongs to $\textsc{Space}(n)$.]

(b) Deduce (without any assumptions) that $\mathrm{P} \neq \textsc{Space}(n)$.

[Hint: Remember the space-hierarchy theorem!]

3. In this problem, you are asked to find the errors in two purported proofs of major "results." In each case, you just need to identify *one* major flaw.

(a) Explain *carefully* what is wrong with the following fallacious "proof" that $\mathrm{P} \neq \mathrm{NP}$:

> Proof by contradiction. Assume that $\mathrm{P} = \mathrm{NP}$. Then $\textsc{Sat} \in \mathrm{P}$, and hence $\textsc{Sat} \in \textsc{Time}(n^k)$ for some k. Because every language in NP is polynomial time reducible to $\textsc{Sat}$, this implies that $\mathrm{NP} \subseteq \textsc{Time}(n^k)$, and therefore that $\mathrm{P} \subseteq \textsc{Time}(n^k)$. But by the Time Hierarchy Theorem, $\textsc{Time}(n^{k+1})$ contains a language that is not in $\textsc{Time}(n^k)$, which contradicts our deduction that $\mathrm{P} \subseteq \textsc{Time}(n^k)$. Therefore we have a contradiction, so $\mathrm{P} \neq \mathrm{NP}$.

(b) Explain *carefully* what is wrong with the following fallacious "proof" that $\mathrm{PH} = \mathrm{P}\textsc{Space}$:

> We already know from class that $\mathrm{PH} \subseteq \mathrm{P}\textsc{Space}$, so it suffices to prove that $\mathrm{P}\textsc{Space} \subseteq \mathrm{PH}$. Now we know that any language L in $\mathrm{P}\textsc{Space}$ can be reduced in polynomial time to TQBF, and hence to TQBF_k for some k, where TQBF_k is the language of true quantified boolean formulas with k levels of alternating quantifiers. But TQBF_k belongs to the kth level of PH, and thus is certainly in PH. Therefore L must also belong to PH, since the polynomial time reduction doesn't take us outside PH. Since we've shown that every language in $\mathrm{P}\textsc{Space}$ belongs to PH, we are done.