Fair and Robust Circle Splines

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Abstract

For many applications, such as aesthetic designs or camera paths, nicely rounded, smooth, interpolatory paths – free of cusps and abrupt hairpin turns – are most important. Such curves can be obtained from globally optimized minimum variation curves (MVC) [Moreton and Séquin 1992], but at high computational costs. We present a blending scheme between circles that robustly produces equally good-looking G2-continuous curves through very challenging sets of interpolation points. One basic method produces such curves in the plane, on a sphere, and in 3D space.

1 Introduction

When the constraints permit it, the MVC will produce circular arcs as solutions, since these have zero variation cost. Thus it is natural to use blends between circular arcs to generate the kind of fair curves mentioned above, and several such schemes have been published, e.g. [Szilvasi-Nagy and Vendel 2000]. But even the best of those can produce unwanted hairpin turns (Fig.1a). We have found that this is caused by the positional interpolation between corresponding arc points, but that it can be overcome, if a suitably chosen angle-based parameterization is used instead.

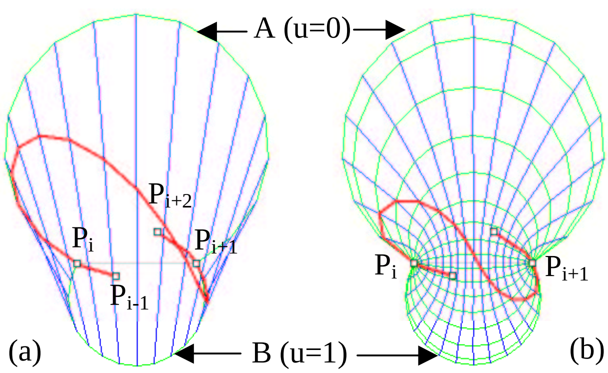

Figure 1. Blended circle spline segment using (a) positional interpolation, and (b) angle-based parameterization.

2 Construction

Given a sequence of constraint points P_0, P_1, ... P_i, ... P_n to be interpolated (the "control polygon"), we form a blend between two circular arcs for every segment (P_i, P_{i+1}). The first arc (A) is defined to go through points P_{i-1}, P_i, P_{i+1} in sequence, and the second one (B) through points P_i, P_{i+1}, P_{i+2}. These two "base arcs" define the tangent vectors t_i and t_{i+1} and the curvatures of the composite curve at points P_i and P_{i+1}, respectively. Our approach guarantees, that the blend curve picks up these end conditions at points P_i, P_{i+1} and that it is well-behaved in between, i.e., has no cusps and no self-intersections, and finite curvature, as long as the control polygon does not have a joint with a turning angle of 180°.

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Figure 1b shows the construction in the plane. The blend between the top (A) and bottom (B) arcs does not occur by simply interpolating circle point positions, as is the case in Figure 1a. Instead, as the point P(u) travels across an arc from P_i to P_{i+1}, the arc morphs from A (u=0) to B (u=1). Any intermediate Arc(u) is defined by the two points P_i and P_{i+1} and by its tangent t(u) at P_i. The direction angle $\tau(u)$ of this tangent is used to parameterize the morphing process of these arcs. It performs a trigonometric blend between the two extreme directions τ_i and τ_{i+1} given by the tangent vectors t_i and t_{i+1} of the base arcs A and B:

$$\tau(u) = \tau_i \cos^2(u\,\pi/2) + \tau_{i+1} \sin^2(u\,\pi/2). \qquad (1)$$

To handle robustly the case of arcs of arbitrary large radii, including straight-line connections between P_i and P_{i+1}, the point P(u) traveling on Arc(u) is parametrically described as a distance $f(u) = |P_i, P_{i+1}| \sin(u\ \tau(u)) / \sin(\tau(u))$ from endpoint P_i and a deviation angle $\phi(u) = (1\text{-}u)\ \tau(u)$ from line segment (P_i, P_{i+1}). This blending scheme guarantees that the parameter lines of constant u do not cross each other (Fig.1b), thereby preventing the blend curve from creating cusps or loops.

If the four points involved in the construction of one blend segment do not lie in a plane, then the blend operation also causes the plane p(u) that contains Arc(u) to swivel around line segment (P_i, P_{i+1}). This happens automatically as the tangent vector t(u) rotates according to Eqn(1) in the plane defined by its two extreme positions t_i and t_{i+1}. Thus the blend curve segment will automatically lie on the sphere that passes through the four defining points. If all points of the control polygon lie on the same sphere, then the whole composite curve will lie on this sphere (Fig.2a). If the control points lie in arbitrary positions in 3D space, space, a nice, fair 3D curve results (Fig.2b).

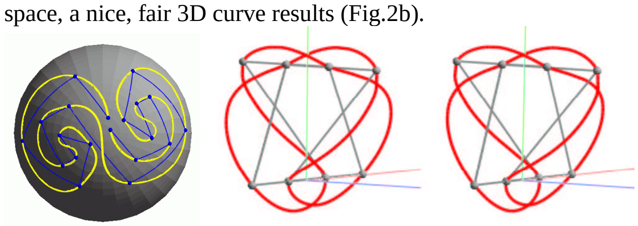

Figure 2. Angle-interpolated circle splines (a) on a sphere, and (b) in unconstrained 3D space (cross-eye stereo view).

The resulting curves have local support and exhibit linear and circular precision by construction, and in general show MVC-like behavior. They are G2-continuous, and with a simple re-parametrization based on arc-length, can also be made C2-continuous. They preserve all symmetries exhibited by the original set of points, including "front-to-back" symmetry, i.e., the curve is not dependent on the direction of evaluation, unlike some quaternion splines.

References

MORETON, H.P. AND SEQUIN, C.H. 1992. Functional Optimization for Fair Surface Design. Proc. ACM SIGGRAPH'92, 167-176.

SZILVASI-NAGY, M. AND VENDEL, T.P. 2000. Generating Curves and Swept Surfaces by Blended Circles. CAGD 17, 197-206.