

## CONGRUENT HAMILTONIAN CYCLES ON THE EDGES OF THE 24-CELL

Carlo H. Séquin

*Address: EECS Computer Science Division, University of California Berkeley, CA 94720.*  
*E-mail: [sequin@cs.berkeley.edu](mailto:sequin@cs.berkeley.edu) URL: <http://www.cs.berkeley.edu/~sequin/>*

In four dimensions there exist six regular polytopes [1]. Many different symmetrical edge-projections from 4D to a 3D subspace have been discussed and illustrated in Volume 1 of Polyhedra [2]. For our studies of edge-colourings of such 3D projections of these polytopes we focus on close-up perspective cell-first projections, where the whole 3D image is completely contained within a single outer cell. These projections maintain a high degree of symmetry and have no coinciding vertices or edges.

We have launched an effort to colour all the edges of each such projection with a set of congruent Hamiltonian cycles. We have already found solutions for the three simpler polytopes (Fig.1) as well as for the 24-Cell, and have developed a strategy that should yield a solution also for the two complex polytopes with 120 and 600 cells, respectively.

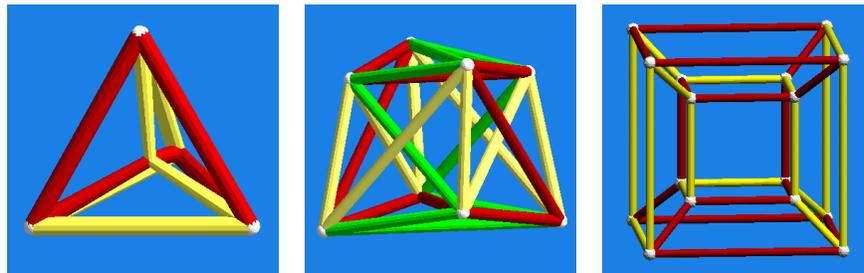


Figure 1: Congruent Hamiltonian cycles on the projections of the three simplest 4D polytopes.

Finding Hamiltonian cycles is a well-known NP-hard problem in graph theory, and the exponential increase in "run-time" has been born out in our non-computer-assisted,

ad hoc, trial-and-error search method for a desired solution. For the 3 simpler polytopes, only a few hours were needed to find solutions for all three of them. But the 24-Cell kept us busy for several days and required a more structured approach for finding a solution.

The graph of the 24-Cell has 24 vertices and 96 edges. Four Hamiltonian cycles are needed, since all vertices are of valence 8. A simplistic backtracking algorithm without any concern for symmetry might run for a rather long time. While the desire for a highly symmetrical solution and for all congruent paths makes the task seemingly harder, it also constrains it in a way that helps to find solutions more quickly – if they exist.

Actually, we found several satisfactory solutions. The first one uses one of the  $C_4$ -axes through an outer octahedron vertex for transforming a primary Hamiltonian cycle into four rotated copies of itself, which can then all be coloured differently. For the black-and-white rendering, we have rendered one path very brightly and one rather darkly; the  $C_4$ -axis points towards the viewer (Fig.2).

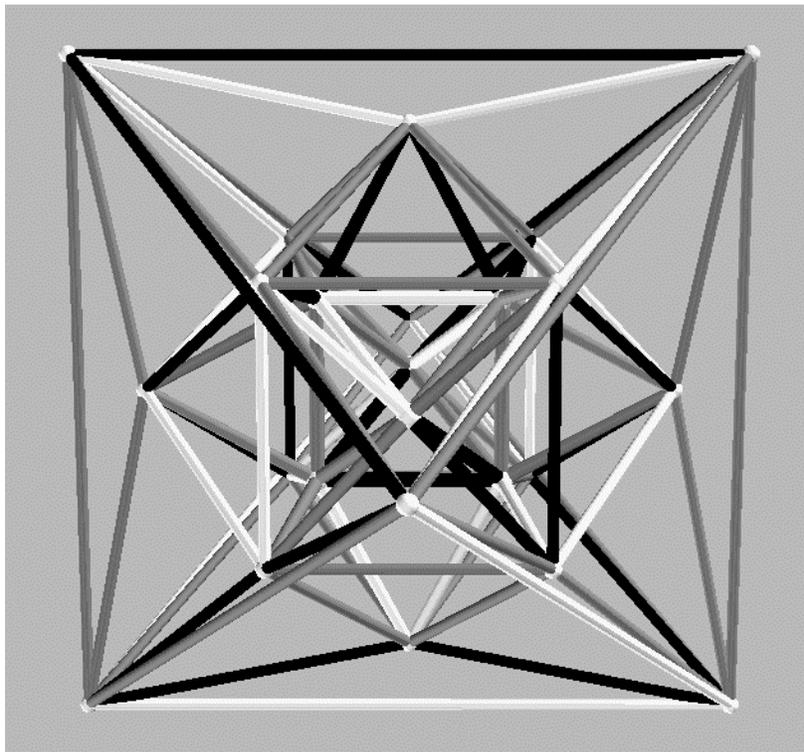


Figure 2: Congruent Hamiltonian cycles on the 24-Cell: 4-fold rotational permutation.

What made the problem tractable was to look at the polytope projection as a set of five nested shells. At the core is an octahedron, which is lined up with the outermost octahedron. In between lies a shell in the form of a cuboctahedron, and then there are two connector shells, containing sets of edges that connect the cuboctahedral edge-frame to the inner and outer octahedra, respectively.

Clearly, each individual shell must adhere to the chosen overall symmetry group. Thus we can first look for suitable colourings for all five shells separately, trying to find patterns with as much symmetry as possible. Then we can try to combine such solutions with the goal of establishing a single closed Hamiltonian cycle for all edges of the same colour. The solution in Figure 2 has been found with only a modest trial-and-error effort.

We then tried to enhance the symmetry of the Hamiltonian paths. First we aimed for  $C_2$ -symmetry for each path, but soon realized that such paths cannot exist. Any possible  $C_2$ -axis intersects the given wire frame in more than just two vertices; thus the final path, which must visit all vertices, would either have self-intersections or break up into more than one cycle. Any for  $C_2$ -symmetry has to be broken to reconnect these shorter loops into a single cycle.

On the other hand, it is possible to have cycles with  $C_3$ -symmetry, since those axes go through the “faces” of the polytope projection. Unfortunately, the desired  $C_3$ -symmetry combined with the overall four-fold colour permutation, forces the Hamiltonian paths to close after visiting one octahedral and the cuboctahedral shell. However, an octahedron also has tetrahedral symmetry as a subgroup. We can associate four different colours with the faces of a tetrahedron and use the symmetries expressed in this colour scheme to define the transformations that map one Hamiltonian cycle into a congruent cycle of a different colour. The result is shown in Figure 3; the dark path displays the 3-fold rotational symmetry. A rotation of the whole wire frame around such a  $C_3$ -axis, leaves one colour in place and cyclically permutes the other three. There are also three  $C_3$ -axes through the vertices of the octahedral shells; a rotation of  $180^\circ$  around such an axis exchanges two pairs of colours simultaneously. Finally, the tetrahedral colouring scheme (Fig.3) also has an in-out-symmetry corresponding to the central point symmetry of the 4D polytope. In 3D, the cuboctahedral shell directly reflects that point symmetry. For all the other edges, there is a corresponding edge of the same colour diagonally across the centre; for the outermost shell, that edges lies on the innermost shell, and for the outer connector shell the matching edge lies on the inner connector shell.

The shell-based search for a Hamiltonian path, with careful consideration of possible symmetries, dramatically reduces the needed computational effort. Instead of searching through a back-tracking tree with 22 7-way choices, we just need to examine at a few mutual orientations of pre-coloured sets of edges on the five nested shells.

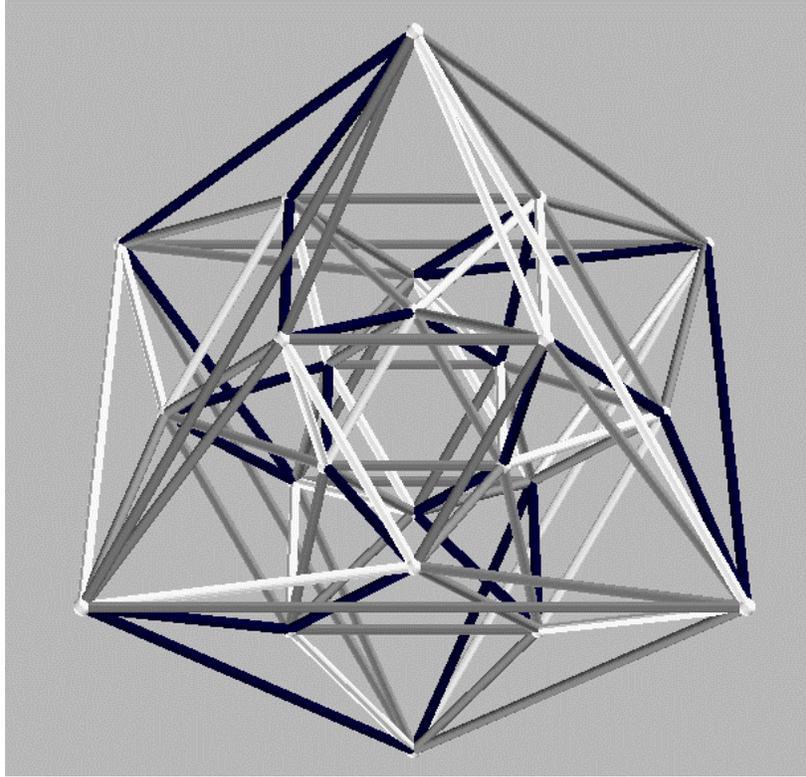


Figure 3: Congruent Hamiltonian cycles on the 24-Cell: tetrahedral permutation.

A similar shell-based search method is now being applied towards the projections of the 120- and 600-Cell polytopes to find two and six congruent Hamiltonian cycles, respectively. A more detailed report on these efforts can be found in the proceedings of the 2004 Bridges conference to be held in Winfield Kansas in July 2004.

Coxeter, H. S. M. (1963), *Regular Polytopes*. McMillan Co. N.Y.

Séquin C. H. (2000), "*Symmetrical Projections of the High-Dimensional Regular Polytopes*," Special Issue of *Symmetry*, Vol 11, No 1-4 pp 65-84: "Polyhedra in Culture and Science", Sándor Kabai, editor.