Tangled Knots

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Abstract

This presentation investigates the role of knots as building blocks for regular 3D structures, such as interlocking instances, spherical objects with polyhedral symmetry, or infinite space-filling lattices. The artistic potential of such configurations is explored through computer graphics imagery and through small maquettes made on rapid prototyping machines.

1. Introduction

Knots fascinate many people, including sailors, cowboys, sculptors, and mathematicians. To the latter, knots are the topologically different configurations that a closed 1-manifold loop can assume. A rough ordering of their complexity is obtained from their crossing number, i.e., the minimal number of self-intersections of the line that must occur when the knot is drawn onto a plane [1]. The simplest knot is the trefoil – a knot with three crossings. The next more complicated knot is the 4-crossing figure-8 knot. Several artists have turned these simple knots into impressive constructivist sculptures.

For the January 2005 snow sculpting competition held in Breckenridge CO, I had designed a sculpture in the form of a twisted Moebius band knotted into an asymmetrical trefoil knot, tailored to fit the 10x10x12-foot size of the snow blocks made available to the contestants. During the first two days of the competition, our team (Team Minnesota, USA; started by Stan Wagon in 1999) sculpted such a large knotted Moebius band in its rough form, removing more than half of the 20 tons of snow contained in the original block. In the remaining two days all three free-standing lobes of this knot were split lengthwise and sculpted into two attractive strands with crescent-shaped cross sections. This split did not really divide the knot into two parts, but instead, because of the twist inherit to a Moebius band (Fig.1a), resulted in a single strand of twice the length of the original band. This strand then formed a much more complicated knot – hence the title for this sculpture: "Knot Divided" (Fig.1b).

Figure 1: *(a) Split Moebius band; (b) our 2005 snow sculpture "Knot Divided".*

2. Split Knots

In this context I was contemplating what would happen if one started from a minimally twisted trefoil band, so that the knot would be cut apart into two entangled trefoil knots. How much could these knots be moved apart from one another – depending on their exact geometry and on the rigidity of the material that they were made of? Thus, after my return to Berkeley, I started to design and fabricate various split trefoil knots – in virtual form as well as on our rapid prototyping machines. Figure 2a shows two trefoil strands that were fabricated lying side by side. Though strongly entangled topologically, the loose geometry allows them to be moved apart by quite some distance. Figure 2b shows the same entanglement of two adjacent trefoils, but pulled apart as far as they can move. This interlocking pattern can then be repeated, resulting in an arbitrary long chain of interlocking trefoils. This iterated sweep movement need not occur along a straight line; it may just as well follow a convoluted, possibly closed, and perhaps knotted path.

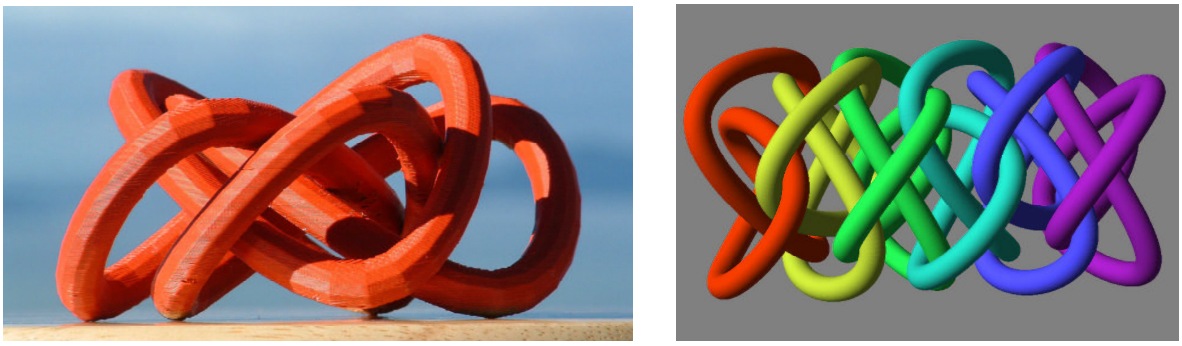


Figure 2: *(a) Split trefoil; (b) expanded interlocking trefoil chain.*

Figure 3 examines the case of a trefoil knot split into 3 strands. Several options exist. We can split the given band into three copies lying on top of each other (Fig.3a); the resulting trefoils are congruent, but their positions are distinct: there are two outer ones and one in the center of the sandwich structure. To avoid such distinct positions, one might prefer to split the original tube into three strands that lie in a triangular arrangement. Figure 3b maintains full 3-fold symmetry for each of the three components, which however differ in size. By applying a different amount of twist during the cutting operation, the knot can be split into three congruent components (Fig.3c); but these no longer exhibit 3-fold symmetry. In all these cases, the entanglement is pretty strong, and the movements of the individual components is thus rather limited.

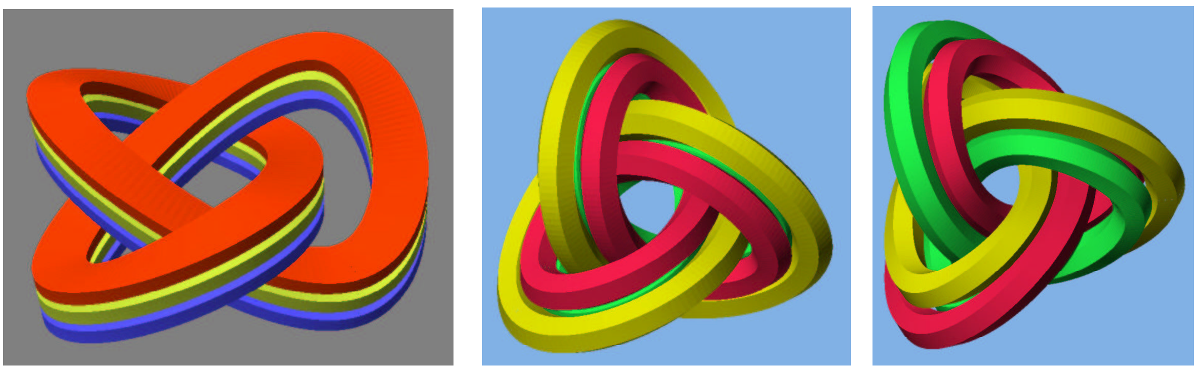


Figure 3: *Triply split trefoils: (a) stack of three; (b) 3-fold symmetric parts; (c) three congruent parts.*

3. Linked Knots

The above examples entice us to explore other ways in which simple trefoil knots can be used as construction elements for judiciously entangled configurations. A trefoil knot can be constructed so that it is relatively flat and of roughly triangular shape. Thus we can start with a polyhedron made from regular triangular faces and try to replace its faces with trefoil knots that interlock along the edges shared by two adjacent triangles. In particular, four trefoil knots can be joined in a tetrahedral formation (Fig.4a), eight knots can form an octahedral shape, and twenty knots can make an "Arabic Icosahedron" (Fig.4b) which I first depicted in virtual form in 1983 [4]. The linking between adjacent trefoils leaves some freedom to the designer: The two trefoils can interlock with just one lobe each, as is the case in the tetrahedral tangle shown in Figure 4a. Alternatively they can link with two lobes each as exemplified in Figure 4b. The amount of warping associated with each of these linkages is also an important design variable.

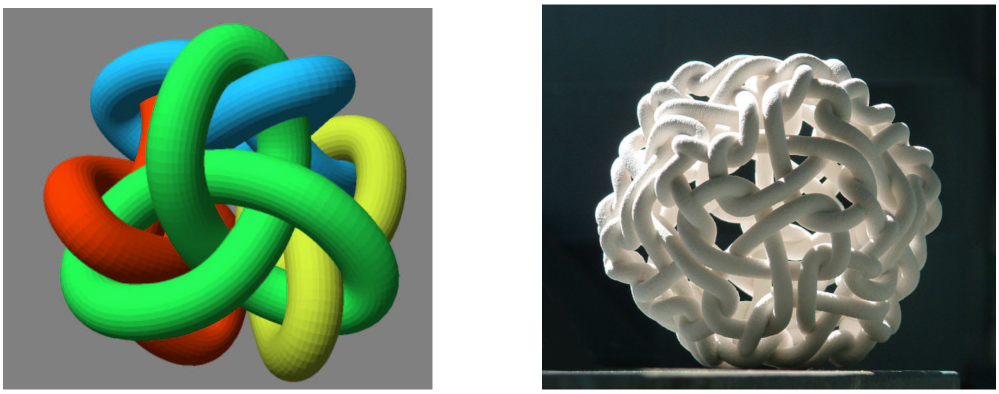


Figure 4: *(a) Trefoil tetra tangle; (b) "Arabic Icosahedron".*

4. Knot Lattices

After exploring the symmetries of the Platonic solids, I was encouraged to study 3D lattices composed of infinitely many identical knot components. Because the linkage should occur truly in three dimensions, a knot with more than three lobes is preferred. I first considered this issue in 1981. Stimulated by a talk by Sculptor Frank Smullin [5], I conceived of the "Granny Knot Lattice" (Fig.5a) [3].

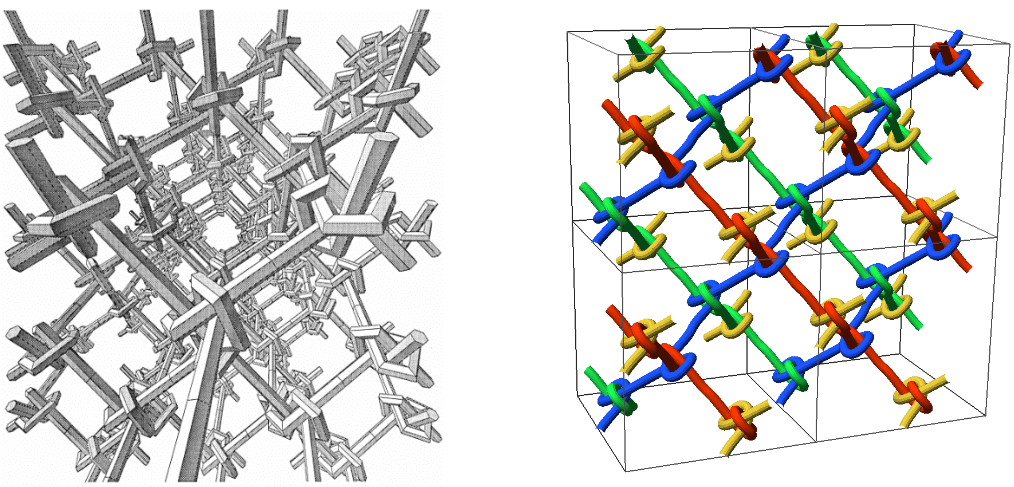


Figure 5: *"Granny Knot Lattice": (a) perspective projection; (b) the connected strands.*

This arrangement corresponds to a diamond lattice. At each atomic site we find a “Granny Knot” with four links sticking out in roughly tetrahedral fashion, similar to the bonds in a carbon. However, this configuration is composed of infinitely long strands that weave their way across the whole lattice as can be seen from the display of just four diamond cells in Figure 5b.

Adhering to the paradigm of using small building blocks in the form of individual knots, I prefer the figure-8 knot, which can also be shaped to have four lobes sticking out in the four tetrahedral directions, and thus can also be assembled into a diamond lattice by linking such lobes pair wise. A much denser arrangement can be obtained if four knots are mutually interlocked around a common point as shown in Figure 6. In either case these knots can be arranged to form an infinite cubic lattice.

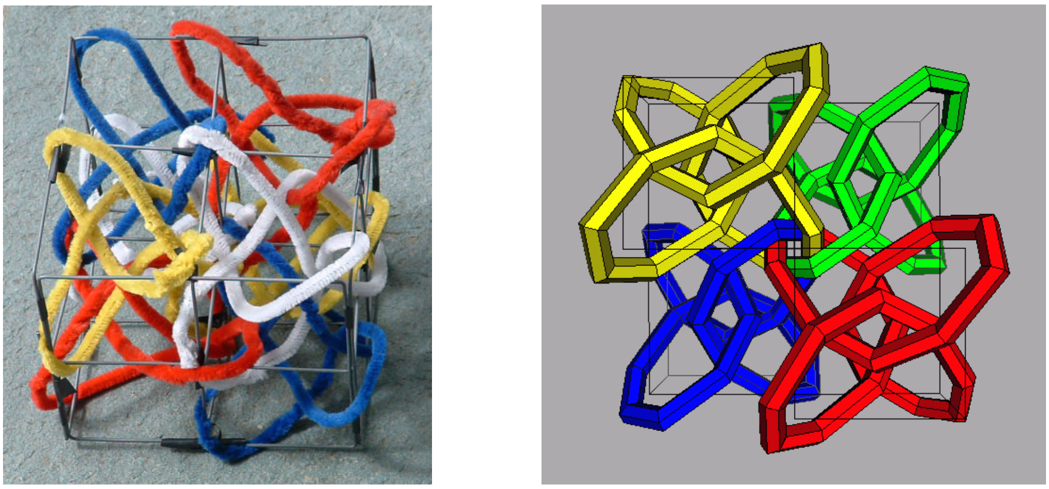


Figure 6: *Figure-8 knots in a cubic lattice: (a) pipe cleaner model; (b) virtual model of lattice cell.*

5. Recursive Knots

Another approach to building complex constellations quickly employs a recursive procedure such as used in the construction of self-similar fractals [2]. An example with nearly planar, 2.5-dimensional Celtic knots is shown in Figure 7a. We apply a simple procedure to replace each crossing of two strands with two S-shapes with a total of nine crossings between them. Very quickly we obtain an intriguing, plane-filling pattern. In Figure 7b we apply a similar procedure to a cubistic realization of a trefoil knot: Every one of its nine 90° corner turns is replaced with a suitably oriented overhand knot. The process can then be repeated at the corners of this configuration.

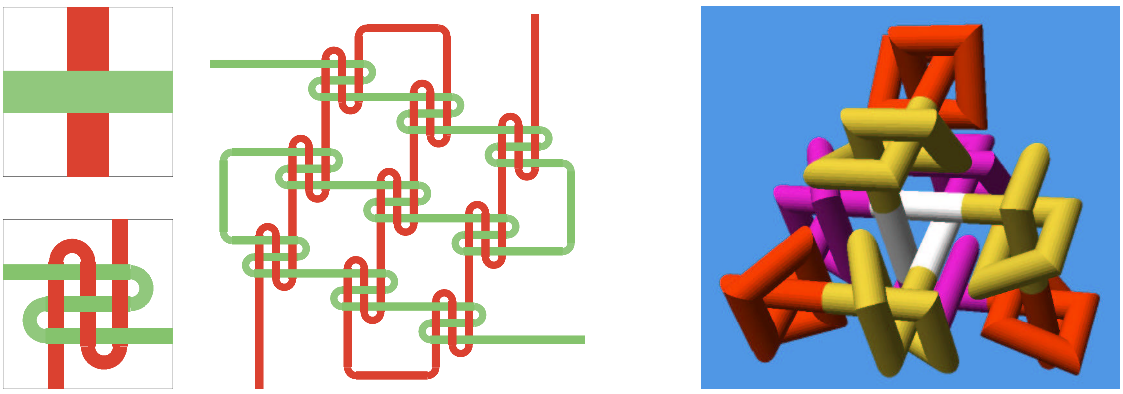
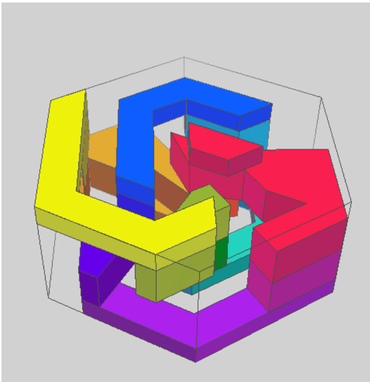


Figure 7: *Recursive knots: (a) substitution steps for a Celtic knot; (b) recursive 3D trefoil knot.*

6. Space-Filling Knots

The examples above lead to the question whether simple knots can be used to tile space without voids. The question was raised by Ian Stewart in his Mathematical Recreations column [6], and a rather complex and convoluted solution was presented, showing how a cube can be partitioned into four trefoil knots. Of course, those cubes can then easily tile all of 3D space. A more natural and less complex solution, where three congruent trefoils solidly fill a hexagonal prism, is shown in Figure 8a; only one trefoil is shown – the other two would be inserted after rotations of 120° and 240°, respectively.

However, breaking space into these compact modular clumps is not truly in the spirit of interlocking knots. I have found other ways of tiling 3D space with shapes that interlock in all directions and across all borders of a convex unit cell; the simplest one, shown in Figure 8b, uses the “unknot”, i.e., a simple toroidal tile.



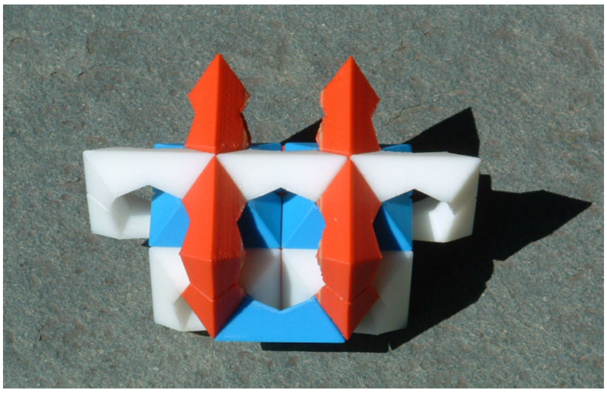


Figure 8: *Tiling 3D space: (a) a simple, “modular” way; (b) a more strongly linked configuration.*

7. Conclusions

The previous sections have shown several methods by which pleasing visual complexity can be created based on constructions with knots. Hopefully this little catalog of possible approaches will stimulate other artists and mathematicians to explore this field and to come up with attractive large-scale sculptures.

Acknowledgements

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