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SYMMETRICAL IMMERSIONS OF LOW-GENUS NON-ORIENTABLE REGULAR MAPS

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*C. H. Séquin, "Symmetrical Hamiltonian Manifolds on Regular 3D and 4d Polytopes" Coxeter Day, Banff, Canada, Aug.3, 2005, pp 463-472.
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*C. H. Séquin, "Intricate Isohedral Tilings of 3D Euclidean Space," Bridges Conference, Leeuwarden, The Netherlands, July 24-28, 2008, pp 139-148.
*M. Howison and C. H. Séquin, "CAD Tools for the Construction of 3D Escher Tiles," Computer-Aided Design and Applications, Vol 6, No 6, pp 737-748, 2009.
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Abstract: Immersions that maximize the displayed symmetry of reflexive regular maps on non-orientable low-genus 2-manifolds are presented in virtual form as well as in the form of simple paper models.

Keywords: Regular maps, non-orientable surfaces, projective plane, cross cap.

1. INTRODUCTION

This is an extension of the work of three years ago (Séquin, 2010) in which symmetrical models of regular maps embedded in orientable surfaces of genus 0 through genus 5 were presented. *Regular maps* are generated by networks of edges and vertices embedded in a closed 2-manifold of a given genus, when all vertices, edges, and facets are topologically indistinguishable and exhibit so-called *flag-transitive* symmetry (Coxeter and Moser, 1980). The most familiar examples are the five Platonic solids, which represent such maps on surfaces of genus zero and which exhibit geometrical symmetry among all their various features.

To apply the concept of regular maps to surfaces of higher genus, we may assume that the network of edges is drawn with marker pen onto an infinitely stretchable thin covering of the whole surface. The crucial requirement then is that one can take any one of all the topologically identical facets and move it to any other facet, in any desired orientation, and the whole network can still be arranged to cover the original image.

It has been known for some time how many such maps can exist on orientable as well as on non-orientable 2-manifolds (Conder and Dobcsányi, 2001; Conder, 2006). Conder's list (as of 2012) lists 3260 non-orientable maps on surfaces of genus 2 to genus 602. The effort to make symmetrical visualization models is more recent. A large number of virtual low-genus models of orientable maps have been presented by (van Wijk, 2009) and by (Séquin, 2010), who also constructed physical models for several of them (Séquin, 2009). In this paper this effort is extended to non-orientable surfaces of low genus. The simplest such surface is the projective plane with a genus of one. Several compact models with reasonably high symmetry have been developed for this surface (Fig.1), and these are the obvious candidates for displaying regular maps on single-sided surfaces of genus one.

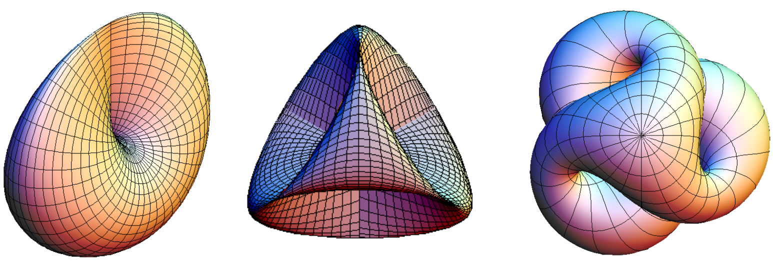

Figure 1: Compact models of the projective plane: (a) Cross surface, (b) Steiner surface, (c) Boy surface.

2. REGULAR MAPS ON THE PROJECTIVE PLANE

Let's first look at "Platonic polyhedra" on the projective plane. To try to form such regular maps, we can take each of the Platonic solids and cut it in half along an "equatorial" circuit of edges and then identify opposite points on the resulting rim. Nothing useful results from cutting the tetrahedron; but for the other four Platonic solids we obtain: the hemi-cube (3 quads), the hemi-octahedron (4 triangles), the hemi-dodecahedron (6 pentagons), and the hemi-icosahedron (10 triangles).

2.1. Paper Models of the Hemi-cube and the Hemi-octahedron

The hemi-cube and hemi-octahedron are duals of one another and exhibit tetrahedral symmetry. These regular maps can thus readily be drawn onto Steiner's Roman surface (Fig.1b), which has the same symmetry. We present the resulting solution with two simple paper models, also showing the cut-out template for each of them in Figure 2. The hemi-octahedron needs four copies of a folded-up triangular face, which are then assembled with tetrahedral symmetry. The hemi-cube consists of three folded-up square faces which need to penetrate each other; the templates show the needed slots.

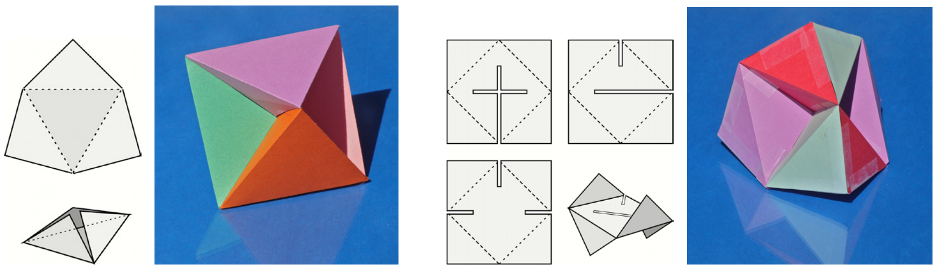

Figure 2: Templates and assembled paper models for: (a) hemi-octahedron and (b) hemi-cube.

2.2. Models of the Hemi-dodecahedron and the Hemi-icosahedron

The Hemi-dodecahedron and hemi-icosahedron are also duals of one another, and both also exhibit tetrahedral symmetry. We will thus draw these two maps also onto Steiner's surface, but now use a virtual rendering based on texture-mapped Bézier patches (Fig.3). Since tetrahedral symmetry is a subgroup of the icosidodecahedral symmetry, we can start from the mapping of the hemi-cube and subdivide its faces appropriately to obtain the faceting representing a hemi-icosahedron or hemi-dodecahedron. In the first case (Fig.3a,b), each of the three square templates carries two full triangles (c,d) and one third of four other faces (a,b,e,f). In the second case (Fig.3c,d), each template carries parts of four different pentagonal faces (p,q,r,s).

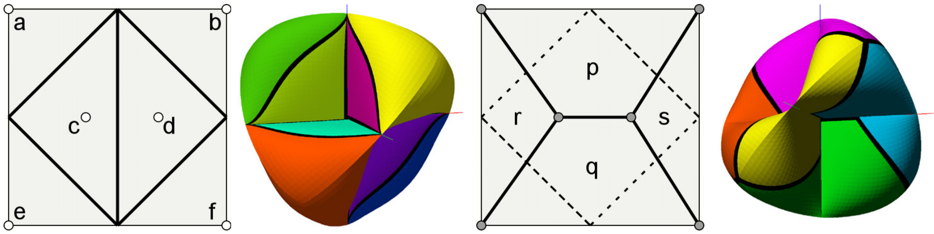

Figure 3: Partitioning of hemi-cube domain to represent: (a) hemi-icosahedron, (b) hemi-dodecahedron.

Steiner's surface is not a smooth immersion of the projective plane; it has six pinch points, called *Whitney's umbrellas*. The most symmetrical, smooth immersion is Boy's surface (Boy, 1903); it has only C_3 symmetry. Thus, for gaining smoothness, we have to pay the price of lesser symmetry. For the "Platonic solids" this price seems not worthwhile. Alternative models of the hemi-cube and hemi-dodecahedron in the form of tubular sculptures have been presented in (Séquin, 2009, Fig.8).

2.3. Hemi-hosohedra and Hemi-dihedra

There are an infinite number of regular maps based on the hemi-hosohedra and their duals the hemi-dihedra. The first kind has just a single vertex and n edges and n faces, and it should thus be amenable to an immersion with n-fold cyclic symmetry. Indeed, for odd n, we can place the vertex at the pole of an n-tunnel Boy surface and let each of the n dihedral faces start at that vertex, wrap through one of the tunnels, and then approach the vertex again from the opposite side in a flipped-over state. Figure 4a shows such a solution for n=5. For the dual figures, the hemi-dihedra, we place the n vertices on the n arms of the Boy surface and let the n-gon edge zigzag back and forth through the tunnels (Fig.4b). If n is even but has odd prime factors that multiply out to an odd value k, then we can still get k-fold symmetry by employing a k-tunnel Boy surface and run n/k faces along each arm/tunnel of that surface. Figure 4c shows a solution for n=12 and k=3.

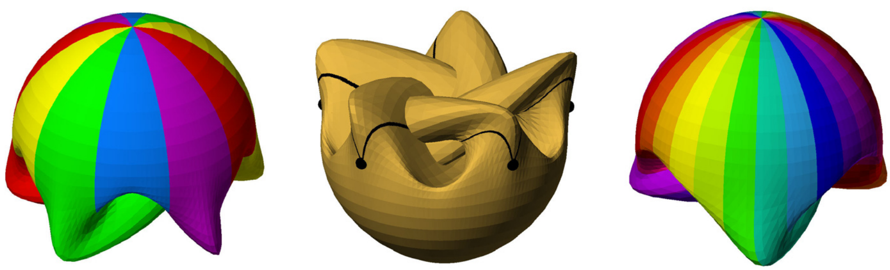

Figure 4: Boy surfaces with: (a) hemi-hosohedron-5, (b) hemi-dihedron-5, (c) hemi-hosohedron-12.

An alternative solution for even-valued n is to draw the map onto a cross-surface (Fig.5a); in this way we get at least 2-fold rotational plus mirror symmetry (Schönflies: C_{2v}; Conway: *22). The case of n=2 warrants some separate discussion, because the hemi-digonal hosohedron is self-dual. It has only one face, one edge, and one vertex. It is best mapped onto a cross-surface. The single vertex should obviously be placed onto the rotational symmetry axis of this surface. When drawing the edge loop onto the cross-surface, we have to be careful to select a path that amounts to a Möbius band when drawn with finite width, and which leaves a single disk when subtracted from the projective plane. Figure 5b shows an appropriate arrangement. If n is a power of 2, $n=2^p$, then we may want to resort to higher-order cross surfaces; now we can achieve full n-fold symmetry for the price of having to tolerate two clusters of higher-order pinch points. A 4-way cross-surface is shown in Figure 5c.

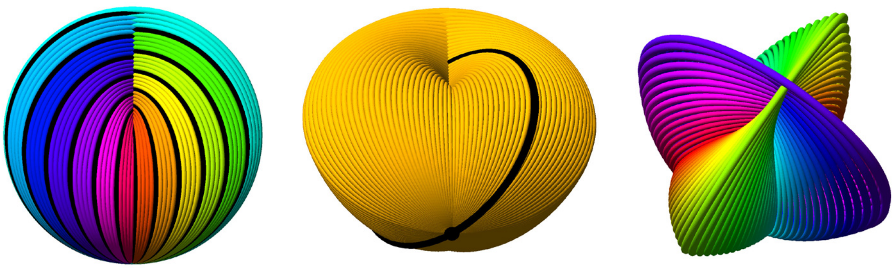

Figure 5: Hemi-hosohedra on cross-surfaces: (a) hemi-hosohedron-12; (b) hemi-digonal hosohedron; (c) hemi-hosohedron-60 on a quad-cross-surface.

3. TECHNIQUES TO MAKE MODELS OF HIGHER GENUS

The projective plane has Euler characteristic χ=1, and genus 1. Logically, we should explore next some non-orientable surfaces of Euler characteristic χ=0, and genus 2, which represent Klein bottles. But it turns out that there exist no regular maps on single-sided surfaces of this genus. This may come as somewhat of a surprise, since on the torus (χ=0, and genus 1) there are infinitely many regular maps! As we proceed to Euler characteristic $\chi = -1$, equivalent to non-orientable genus 3, known as Dyck's surface (Dyck, 1888), there are still no regular maps! The first higher-genus, single-sided, regular maps appear for genus 4. Table 1 lists the salient features of all the non-orientable regular maps from genus 4 to genus 8.

Name	#	facets	#	vertices	# edges	mV	mF	Pet.-L.
N4.1	6	quads	4	6-valent	12	2	2	6
N4.2	6	quads	4	6-valent	12	2	1	3
N5.1	15	quads	12	5-valent	30	1	1	6
N5.2	9	quads	6	6-valent	18	2	1	4
N5.3	6	5-gons	6	5-valent	15	1	1	3
N5.4	3	6-gons	3	6-valent	9	3	3	3
N6.1	20	triangles	6	10-valent	30	2	1	10
N6.2	20	triangles	6	10-valent	30	2	1	5
N6.3	20	quads	16	5-valent	40	1	1	5
N7.1	15	quads	10	6-valent	30	1	1	5
N7.2	9	quads	4	9-valent	18	3	2	9
N8.1	84	triangles	36	6-valent	126	1	1	9

Table 1: List of the non-orientable maps of genus 4 through genus 8.

Wedd, N. S. (2009, 2010) gives nice, abstract, 2-dimensional diagrams describing many regular maps of various kinds, some of them in multiple different ways (Fig. 6). These topological diagrams are very helpful for obtaining quickly an understanding of the connectivity of the various facets in the fundamental domain, and they simplify the somewhat tedious step of extracting that information from Conder's characterization, which relies on *relator* expressions (Conder 2012). The samples chosen also show that there are potentially many different ways to represent these maps, and that it may not be obvious which is the best visualization model.

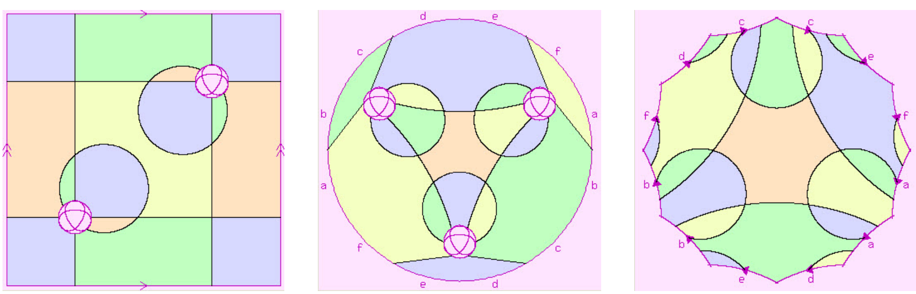

Figure 6: Three topological diagrams for N4.1d by Wedd (2009, 2010).

To construct symmetrical 3D models of non-orientable maps, I then follow the same technique that I used for finding such models on orientable handle-bodies (Séquin, 2010). The first step is to construct a single-sided 2-manifold of the needed genus that has some of the symmetries that are implied by the number of vertices, edges, or faces in the regular map being studied. In general, a single-sided surface of genus g can be constructed by starting with a sphere and grafting onto it g cross caps; each one increases the genus by one. However, since smooth immersion of the regular maps may look more attractive, we prefer to construct 2-manifolds without singularities. Thus, rather than using cross-caps, we prefer to use Boy caps, which are constructed by removing a small disk from a Boy surface. Such Boy caps can be constructed with r-fold rotational symmetry, as long as r is an odd number ≥ 3 (Séquin, 2013). By starting with a Platonic solid and placing appropriate Boy caps or cross-caps on all faces, we can readily obtain non-orientable surfaces of genus 4, 6, 8, 12, and 20, exhibiting maximal symmetry (Fig.7). The Archimedean solids, used in a similar manner, may be the next best starting point.

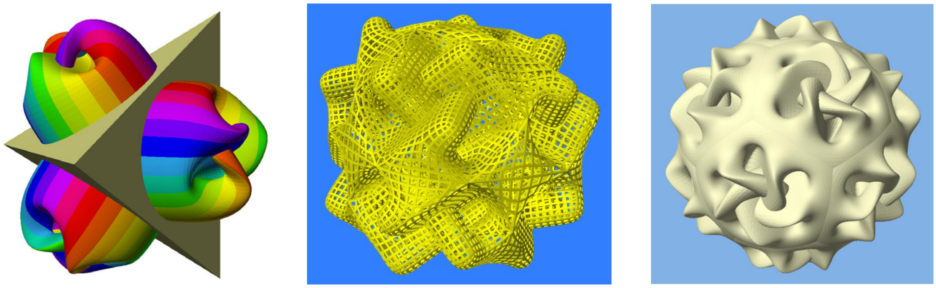

Figure 7: Symmetrical single-sided canvases derived from the Platonic solids: (a) tetrahedral genus-4, (b) octahedral genus-8, (c) dodecahedral genus-12**.**

By starting instead from Steiner's Roman surface (Fig.1b), which has tetrahedral symmetry and is already of genus 1, we can readily obtain non-orientable surfaces of genus 4, 7, 11, 13..., by placing cross-caps or Boy caps onto the four faces or onto the six "edges" of the Steiner surface, where the Whitney umbrellas occur. The resulting "canvasses" all still exhibit tetrahedral symmetry.

Another option is to use *cross handles* (Fig.10a). Such handles connect two punctures in the sphere onto which we could have grafted cross cap; they will therefore increase the genus by two. Since we need fewer of them to build a surface of a desired genus, they will typically lead to surface immersions with lower symmetry. They provide, however, an intriguing option to connect antipodal faces on a Platonic or Archimedean solid, while hiding the pinch points associated with the simplest version of such a cross handle on the inside of the model, and in this mode they will not reduce symmetry.

4. NON-ORIENTABLE REGULAR MAPS OF GENUS 4

For single-sided surfaces of genus 4, Conder (2006) shows the existence of two regular maps consisting of six quadrilaterals joining in four valence-6 vertices. The numbers of vertices (4) and faces (6) remind us of numbers associated with a tetrahedron or a cube. Indeed, the most symmetrical way to realize a non-orientable 2-manifold of genus 4 is to graft four Boy caps of the same chirality in tetrahedral positions onto a sphere; here the 3-fold symmetry of Boy's surface is most welcome! Figure 7a showed a conceptual arrangement of the four Boy surfaes on the faces of a tetrahedron, and Figure 8a shows the unfolded net of this tetrahedron with the Boy caps represented symbolically as rainbow disks. These Boy caps act as switching stations that route an arbitrary number of edges from one side of the disk to the opposite side of that disk without causing any crossings among them. For the regular map N4.2 we just need to route three edges through those Boy caps, and we show them symbolically as curved, entangled lines with three over/under bridges (which are provided by the Boy cap). These diagrams make it easy to trace out the individual edges, to paint in the six facets, and to check the length of the Petri polygon (just remember to switch sides when following an edge through one of the Boy cap switching stations!).

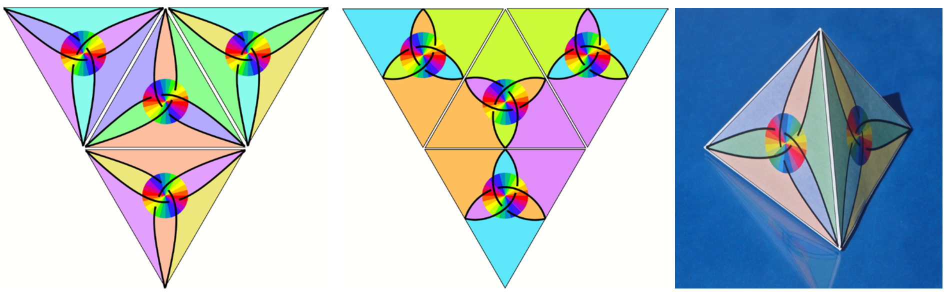

Figure 8: (a) Regular map N4.2, (b) and its dual map N4.2d; (c) folded-up paper model of N4.2.

Figure 8b shows the dual of this map. The six vertices now fall on the edges of the tetrahedron, and each of the four hexagons has one of its corners twisting through one of the Boy caps. Figure 8c is a folded-up paper model of the net shown in Figure 8a. For completeness, we show in Figure 9 some smoother, more organic" surfaces that can serve as single-sided canvasses of genus 4. Figure 9a depicts a simple, fat disk with four cross-cap slots; this is probably the most straight-forward way of depicting such a surface. Figures 9b and 9c show more tightly fused arrangements of four Boy caps; they differ only in the relative rotation angles between adjacent Boy caps.

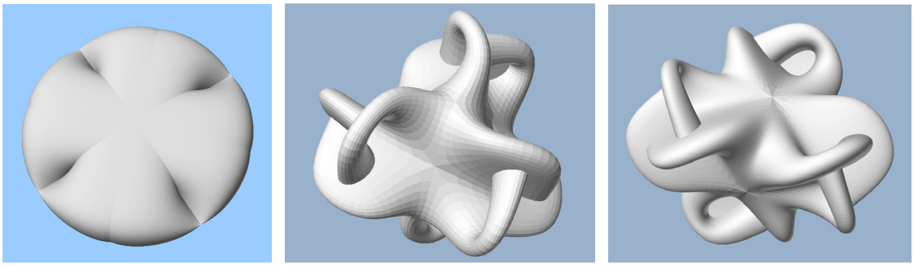

Figure 9: Symmetrical, non-orientable surfaces of genus 4: (a) four cross caps in a "wheel" with D_{4h} symmetry; (b,c) four Boy caps in two different tetrahedral arrangements.

For genus 4 there is a second type of a non-orientable regular map, which also has six quadrilateral faces and four valence-6 vertices. This map, N4.1, differs from the map N4.2 above by its face-multiplicity: each face borders each other face exactly twice. This makes it much trickier to find a good symmetrical immersion in 3D Euclidean space; the tetrahedral arrangement used above just does not work.

I found a nice solution using internal cross-handles (Fig.10a) in a 4-sided prism. In order to maintain symmetry, these tubes will have to intersect one another at the center of the cube; but since any closed non-orientable surface must have some intersections anyhow, this is not a real concern. Now we can let the four hexagonal faces pass in pairs through the two cross-handle tunnels, emerging in reversed orientation on the other side. In Figure 10b I first show a virtual model of the *dual* of this map, N4.1d, since it accommodates a very nice placement of the six valence-four vertices. The folded-out net of this surface is shown in Figure 10c, with the four tunnel entrances shown symbolically as circles with a black-and-white hatching.

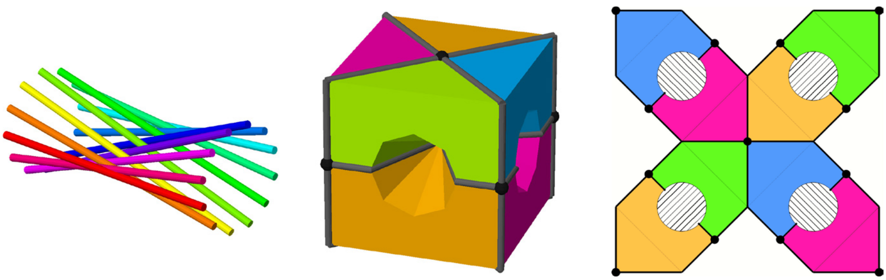

Figure 10: (a) Conceptual view of an internal cross-tunnel; (b) virtual model of the regular map N4.1d; (c) unfolded net for a paper model of the regular map N4.1d.

Now with a suitable single-sided surface model established, we can also draw the actual map N4.1. If we place the four vertices at the centers of the four facets of the map

N4.1d, then they all fall at the center location of the cube (Fig. 11a,b). To make all the edges visible that connect these valence-6 vertices, we move all four vertices to the front faces of the cube, thereby reducing the symmetry of this particular model (Fig. 11c).

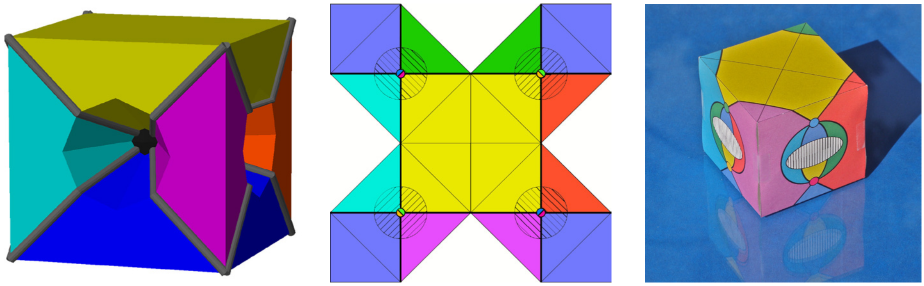

Figure 11: (a) Virtual model of map N4.1; (b) net of N4.1 with highest symmetry with all vertices at the center; (c) folded-up paper model N4.1 with the four vertices moved outside the tunnels.

5. NON-ORIENTABLE REGULAR MAPS OF GENUS 5

Conder's list continues with four maps for genus 5. Let's start with map N5.1. It has 15 quadrilateral faces and twelve valence-5 vertices. As a canvas we use a Steiner surface (Fig.1b), which is already of genus 1, and place four Boy caps in a tetrahedral manner on the four outer bulges. We place the twelve vertices of N5.1 to form three squares at the centers of the three intersecting equatorial planes, and twist each one of the remaining twelve quadrilaterals through one of the Boy caps (Fig.12a).

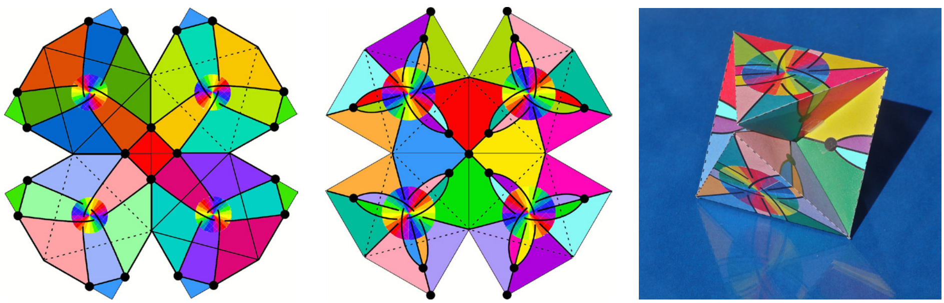

Figure 12: (a) Net of map N5.1, (b) net of its dual map N5.1d, (c) folded-up paper model of N5.1d.

For the dual map, N5.1d, we place groups of five vertices each in a cross-shaped constellation at the centers of the three equatorial planes. The twelve pentagonal faces each have two of their corners extending through two Boy caps (Fig.12b). A folded-up paper model is shown in Figure 12c.

Among the genus-5 maps there are also two self-dual maps. Let's next look at map N5.3, which has six pentagons, six valence-5 vertices, and 15 edges; it is "well-behaved" in the sense that its face-multiplicity is 1. Again we graft four Boy caps onto the four triangular outer bulges of a Steiner surface and place the six vertices at the pinch points of the Whitney umbrellas. Each of the three central equatorial planes gets split into two halves, thereby forming the six required pentagonal facets (Fig.13).

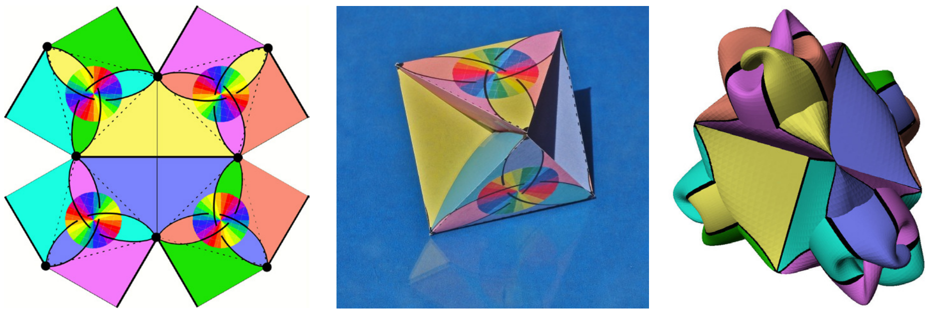

Figure 13: (a) Symbolic net of regular map N5.3; (b) folded-up paper model; (c) virtual rendering using four Boy caps grafted onto a Steiner surface.

Map N5.4 is also self-dual, but it is much more convoluted, since it has vertex- and face-multiplicities of 3. It comprises three hexagonal faces and three valence-6 vertices; every face is connected to the two other faces three times. Because there are so many suggestions of 3-fold symmetry, we will immerse this map onto a torus with three grafted-on cross caps. Figures 14a and 14b show the front and back of a torus that shows this mapping in a symbolic way, representing the cross-caps again as rainbow disks. Figures 14c and 14d then show a virtual model of this immersion with fleshed-out cross-caps attached to a torus.

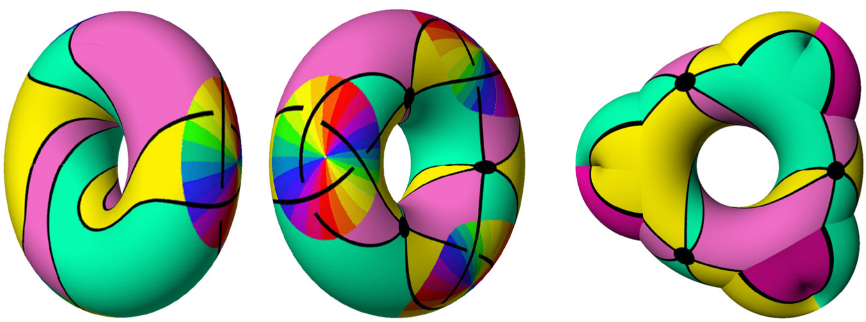

Figure 14: Regular map N5.4: (a) back and (b) front of a symbolic mapping onto a torus; (c) front of a virtual model with fleshed-out, grafted-on cross-caps.

6. NON-ORIENTABLE REGULAR MAPS OF GENUS 6

Among the regular maps of genus 6, map N6.2d is of particular interest in the context of this paper; it is a second good candidate to demonstrate the use of internal cross-handles. Its six decagonal facets suggest the use of a cube-like structure. Its twenty vertices can then be placed conveniently at the eight cube corners and onto the twelve edges. Again, this map is heavily connected: each facet touches each of the other facets twice! Since there exist no 4-way symmetrical Boy caps, we could place a cross-cap on each cube face and let the 2-fold symmetry of the cross cap break the 4-fold symmetries of the cube faces. However, overall we can still orient the cross-cap creases in such a way that the surface has the symmetry of the Borromean rings (Schönflies: T_h; Conway: 3*2).

Alternatively, we can obtain a genus 6-surface with equivalent symmetry by routing three cross-handles of the type shown in Figure 10a through the center of the cube. Now we can let the six decagonal facets pass in pairs through the three intersecting cross-tunnels, emerging in reversed order on the other side. For each visible facet we can see five vertices (Fig.15a); the other five vertices occur in antipodal positions.

We can also embedd map N6.2 into the same surface. For a most symmetrical model the six vertices corresponding to the facets of at N6.2d would again be placed in the middle of the cross handles, but this would hide them from view. Instead we reduce the symmetry of our solution and bring those vertices out onto the cube faces. The resulting pattern is shown in Figures 15b and 15c. The cross-tunnels are symbolicly shown by the narrow ellipses with the black-and-white hatching.

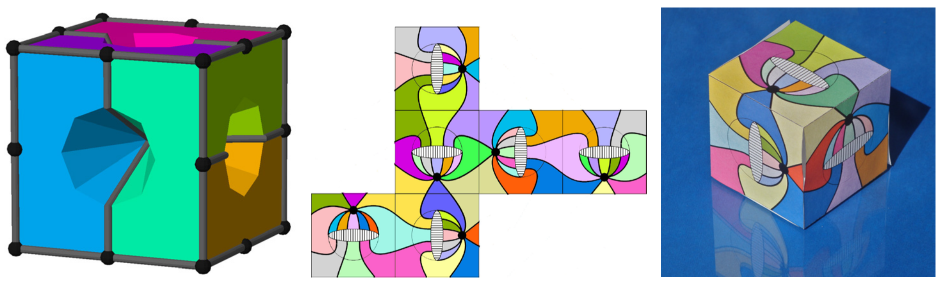

Figure 15: (a) Virtual model of the regular mapN6.2d; (b) folded-out net of the map N6.2; (c) folded-up paper model of the regular map N6.2.

7. NON-ORIENTABLE REGULAR MAPS OF GENUS 7

Conder's list continues with two maps for genus 7. Map N7.1d has 10 hexagonal facets and 15 vertices of valence-6; its face- and vertex-multiplicities are both 1. A suitable canvas seems to be a Steiner surface with six cross-caps added onto the Whitney umbrellas. Here we actually cut the cross-surface along a figure-8 profile line and match this with a similar contour carved from the Steiner surface around the Whitney umbrella (Fig.16a). Each hexagonal facet then extends three alternating corners through three of these cross-caps, and each cross-cap will have to accommodate five of these hexagon-corners (5 edges); from this point of view, we should realize these "switching stations" as 5-fold symmetrical 5-tunnel Boy surfaces. However, since this does not match well with the 2-fold rotational symmetry of the Whitney umbrellas, I rather stick with the simpler cross caps. A vertex has to be placed between every pair of cross-caps; this places one vertex on every octahedral edge, and the three vertices linking antipodal cross-caps at the center of the structure (Fig.16b).

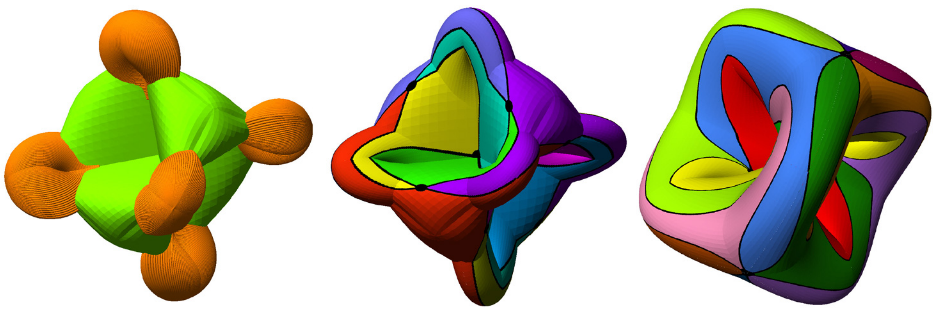

Figure 16: (a) Steiner surface with six cross-caps placed on its Whitney umbrellas; (b) dual map N7.1d; (c) regular map N7.1 on a smoothed-out surface model.

The actual map N7.1, the dual of N7.1d, has 15 quadrilateral facets and 10 vertices of valence-6. It should readily fit onto the same surface. If we place the vertices at the centers of the facets of map N7.1d, then four vertices will end up in the centers of the four outer bulges of the Steiner surface. The other six vertices will land on the half-axes where the three equatorial planes intersect (Fig.15c).

The second map N7.2 has again much stronger connectivity. It has nine quadrilateral faces and four valence-9 vertices; its vertex multiplicity is 3, and its face multiplicity is 2. It seems plausible to try to map it onto the same canvas used above, placing the four vertices at the outer Steiner bulges and centering the nine quadrilateral facets at the six humps of the cross-caps above the Whitney umbrellas and at the three equatorial face-centers. However, this map has so far resisted all my attempts to find a 3D model with tetrahedral symmetry. The problem seems to be that, because of the vertex multiplicity of 3, among the nine edges emerging from each vertex, every third edge must connect to the *same* vertex. This ordering constraint conflicts with the symmetry of a tetrahedral model, where we find three *different* vertices when looking from one corner in three equally spaced, symmetrical directions 120° apart (Fig.17a).

Figure 17: (a) Partial paper model of regular map N7.2, trying to connect the four vertices in a symmetrical manner; (b) chain-link loop of nine Möbius bands showing the topology of dual map N7.2d.

Topologically, the dual map N7.2d can best be understood as chain-link loop of nine Möbius bands. Figure 17b illustrates this connectivity. It shows the edge regions of the four 9-gons in the four primary colors, and also indicates a Petrie polygon of length nine as a black, dashed center line on half the edge-bands. This 9-link chain might suggest that this map should be immersed in a surface with 9-fold prismatic symmetry. However, the nine individual Möbius bands would then imply the presence of nine cross-caps or Boy caps, and that would produce a single-sided surface of genus 9, rather than of the desired genus 7.

8. NON-ORIENTABLE REGULAR MAPS OF GENUS 8

For genus 8 there is only a single regular map. But N8.1 also offers a tough challenge to make a symmetrical 3D model. This map has 84 triangular facets, 36 valence-6 vertices, 126 edges, and Petrie polygons of length 9. None of these numbers has a divisor of 8, so it seems that the highly regular genus-8 surface in Figure 7b may not work. On the other hand, the Petrie length of 9 suggests that in this map there are triangle strips of length 9 twisted into Möbius bands. Such Möbius bands can readily be deformed into Boy caps with a rim in the shape of a regular 9-gon; so let's place eight of those onto the faces of an octahedron. In isolation, these eight bands would comprise 8*9*2*2=144 edges. By sharing 2/3 of the rim edges of these cross caps with a neighbour, we save 24 edges overall. So now we are 6 edges short of the desired total of 126 This may imply that at each of the six corners of the octahedron we need to split a quadrilateral opening with one additional edge; and this would explain why we will not get full octahedral symmetry. However, I have not yet been able to place all the edges onto this “Octa-Boy” surface so that a highly symmetrical model of the regular map N8.1 emerges.

Another plausible approach would focus on the divisors 6 and 7, which show up prominently in this map. Since I cannot see any way how to place seven cross-caps onto a genus-1 non-orientable surface, the best attempt might come from placing six cross-caps onto a torus in the spirit of the solution for map N5.4. Now the question arises: will all the triangles have to be routed through one of the cross caps, or will it be sufficient to twist only a subset of them? So far I have not been able to figure out how to split the set of all faces into two sub-populations of twisted and untwisted triangles in a way that is compatible with the symmetry of the envisioned model.

9. CONCLUSIONS

Finding symmetrical realizations in 3D space for some of the orientable regular maps was quite difficult (Sequin 2010). To construct such visualizations for non-orientable maps is even more challenging. This paper gives a glimpse of my somewhat ad-hoc, intuition-driven approach of finding highly symmetrical immersions of low-genus, single-sided regular maps. Already in my search for 3D models of orientable regular maps, I had made an attempt to use a computer program to help me find symmetrical solutions for a particular map. The approach I pursued was somewhat complementary to the work by van Wijk (2009), which creates symmetrical configurations in a procedural manner and then checks which one of them happen to represent regular maps. Alternatively, with the help of a couple of undergraduate students, I tried to write a program that would do a backtracking search with the goal of placing all the map edges crossing-free on a given surface, onto which all vertices already had been placed manually in a plausible, symmetrical pattern. But this initial attempt was not successful. To develop a similar program for non-orientable surfaces would probably be even more challenging. Thus, for the moment, I have to rely on intuitive guesses and on much trial and error to find feasible routings for all the edges of a given map.

Nevertheless, this intuition-driven approach has produced some intriguing models, which hopefully can convey a good impression what such non-orientable regular maps look like. Among all the maps listed in Table 1, there are still some hold-outs for which

I have I have not yet found a satisfactory 3D model. Moreover, with all the models depicted the natural question arises, whether these models are truly of the highest possible symmetry achievable in 3D space. This is mostly an open question. I am in no position to make any such formal claim. All these models fall far short of the intrinsic topological symmetry of the corresponding maps, which is typically on the order of twice the number of edges in the map. But the possibilities are rather constrained for designing surfaces of high symmetry with the required genus in 3D space. If we can then draw the given map in a symmetrical manner onto such a surface model, we can be reasonable sure that we have done what is best possible.

Some of the displayed artefacts are not only useful mathematical visualization models, but they may constitute attractive art-objects in their own right. At an appropriate scale, and painted with a carefully chosen palette of colors, they might make attractive constructivist sculptures.

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