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Star Theorem Patterns Relating to $2n$-gons in Pascal's Triangle — and More

Peter Hilton

Mathematical Sciences Department, SUNY Binghamton Binghamton, NY 13902-6000

Jean Pedersen

Department of Mathematics & Computer Science, Santa Clara University Santa Clara, California 95053-0290

Email: jpedersen@scu.edu

Carlo H. Séquin

EECS Computer Science, University of California, Berkeley, California 94720-1776

Email: sequin@cs.berkeley.edu

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Abstract. We first pose a Sudoku-type puzzle, involving lattice points in Pascal's Triangle, and lines passing through them. Then, through the use of an expanded notation for the binomial coefficient $\binom{n}{r}$, we exploit the geometry of Pascal's Triangle and produce a non-computational solution to the well-known Star of David pattern about products of two subsets of the 6 nearest neighbors to a given binomial coefficient. We generalize and obtain analogous patterns, with similar geometric proofs, for $2n$-gons where the entries at the lattice points of the symmetric (but not necessarily triangular) arrays are what we call separable functions (e.g., binomial coefficients, Leibnitz Harmonic coefficients, and Gaussian Polynomials, or q-analogues).

Keywords: Binomial coefficients; Multinomial coefficients; Separable functions; Leibnitz harmonic coefficients; Gaussian polynomials; q-analogues; Higher order separable functions; Star of David theorems; Stop-sign theorems; $2n$-gon theorems.

1. A New Type of Geometrical Placement Problem

There is strong evidence that people like symmetry and orderly patterns. The current popularity of the Sudoku puzzles is a manifestation of this craving. The patterns relating to complete symbols have been discussed and described by P. Hilton, J. Perdersen and others in [7]. In this article we look for patterns among the numbers in Pascal's Triangle and relationships between them, but it turns out that we gain much more than that. At a very basic level one can think of this topic as being a puzzle, or game, that is very much like Sudoku. However, this game is played on an array of points arranged in a triangular lattice, or 3-line grid (as opposed to regions arranged in a square, or 2-line, grid), and the object is to locate points that satisfy geometric constraints (rather than finding numbers that satisfy arithmetic constraints). Viewed this way our "game" can be played without even knowing that an important underlying significance of finding these points is that you get startling results about the products of very large numbers in Pascal's Triangle. Surprisingly, the biggest pay-off of all is that, having found lattice points that satisfy the given rules, mathematics then gives us much more; in the first case — the triangular array of Leibnitz harmonic coefficients — we get results involving complicated-looking rational numbers, and in the second case — the Gaussian polynomials, or q-analogues — we get results that can involve very large polynomials.

So, if you simply want to play a game like Sudoku, the straightforward rules we are about to give you will provide you with an endless supply of puzzles. On the other hand, if you want to really get involved with the underlying significance of what the solution to these puzzles tells you mathematically in the Pascal Triangle (or certain other arrays of gridlines), you will be off to a good start.

So here are the rules to our game (the mathematics will come later). In the triangular array (1), that is pretending to be an array of equilateral triangles with entries shown at the vertices, notice that we have replaced six of the lattice points by suit symbols, three $\clubsuit$ and three $\diamondsuit$. This arrangement is particularly pretty in that the boundary of these six symbols forms a regular convex hexagon (or 6-gon). These $\clubsuit$ and $\diamondsuit$ located on the boundary of our convex 6-gon satisfy the following rules:

(a) If you draw a horizontal line through any one of these symbols (a $\clubsuit$ or $\diamondsuit$) then there is exactly one corresponding mate (a $\diamondsuit$ or $\clubsuit$, respectively) on that line.

(b) If you draw an upward sloping line (at 60°) through any one of these symbols (a $\clubsuit$ or $\diamondsuit$) then there is exactly one corresponding mate (a $\diamondsuit$ or $\clubsuit$, respectively) on that line.

(c) If you draw a downward sloping line (at 60°) through any one of these symbols (a $\clubsuit$ or $\diamondsuit$) then there is exactly one corresponding mate (a $\diamondsuit$ or $\clubsuit$, respectively) on that line.

(d) Having drawn the lines mentioned in parts (a), (b) and (c) every $\clubsuit$ and every $\diamondsuit$ has exactly three lines passing through it.

$$\begin{array}{ccccccccccccc}
\bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet \\
 & \bullet & & \bullet & & \clubsuit & & \diamondsuit & & \bullet & & \bullet & \\
\bullet & & \bullet & & \diamondsuit & & \bullet & & \clubsuit & & \bullet & & \bullet \\
 & \bullet & & \bullet & & \clubsuit & & \diamondsuit & & \bullet & & \bullet & \\
\bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet
\end{array} \tag{1}$$

Before proceeding you might like to draw the lines satisfying the conditions (a) through (d) in array (1) and study the result. You should have 9 lines, 3 sets of 3 parallel lines each. Notice, too, that the 6-gon has 6 axes of symmetry.*

Now, here are some puzzles for you:

(i) Can you locate 6 vertices in the triangular lattice so that their boundary forms a convex semi-regular 6-gon (every other edge of the same length) that satisfies the rules (a) through (d)? [Hint: This figure will have only 3 axes of symmetry.]

(ii) Can you locate 8 vertices in the triangular lattice so that their boundary forms a convex 8-gon, that satisfies the rules (a) through (d)? How many axes of symmetry does your solution have? Can you get other solutions with more, or fewer, axes of symmetry? What is the maximum number of axes of symmetry possible for such an 8-gon? What is the smallest configuration?

(iii) Can you locate 10 vertices in the triangular lattice so that . . .

You could, in fact, try this for any even number with varying restrictions.

In the following sections we will discuss what the solution to these little puzzles means mathematically.

2. Setting the Stage With an Expanded Notation for $\binom{n}{r}$

It is common practice to write the coefficient of x^r, in the expansion of the binomial $(1+x)^n$, as $\binom{n}{r}$ with $0 \le r \le n$. Thus $\binom{n}{r}$ is seen to be $\frac{n(n-1)\dots(n-r+1)}{1\cdot 2 \quad \cdots \quad r}$, or equivalently, $\frac{n!}{r!(n-r!)}$. The array containing these numbers is commonly called Pascal's Triangle. Its first six rows look like this:

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array} \tag{2}$$

If you superimpose the part of the array (1), involving the ♣ and ♢, on the set of numbers in array (2) you will see that when the center of array lands on 3 then the product of the ♣ entries is $(2)(1)(6) = 12$, while the product of the ♢ entries is $(1)(4)(3) = 12$. What is really remarkable is that no matter where

*Assuming this is a genuine equilateral triangle lattice, this means there are six straight lines that can be drawn such that if the hexagon were folded on any one of those lines the two halves on either side of that line would coincide with each other. In particular 3 axes of symmetry are formed by lines passing through opposite vertices (when the ♣ vertices coincide with ♣ vertices and the ♢ vertices coincide with ♢ vertices) and 3 axes of symmetry are formed by lines passing through the midpoints of the opposite sides of the hexagon (when the ♣ vertices correspond with ♢ vertices) .

we superimpose that array onto Pascal's Triangle, this property is true. † If you draw the lines connecting the ♣ and the lines connecting the ♢, two intersecting equilateral triangles emerge, and therefore this pattern in the Pascal Triangle is referred to as the

Star of David Theorem. *In array* (1) *if all symbols are located at lattice points in the Pascal Triangle, then* $\prod \clubsuit = \prod \diamondsuit$.

We will give the proof of this theorem after discussing a particularly useful expanded notation for $\binom{n}{r}$ and making an observation about that notation.

Since we are clearly emphasizing the symmetry, and geometry, of Pascal's Triangle, it is natural to think of the binomial coefficient as being the coefficient of $a^s b^r$ in the expansion of $(a+b)^n$ as $\left(\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right)$, with $0 \leq r, s \leq n,\ \ r+s=n$, and $\left(\begin{smallmatrix} & n & \\ 0 & & 0 \end{smallmatrix}\right) = 1$ and $\left(\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right) = 0$ if $r, s > n$. Arithmetically $\left(\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right)$ is just $\frac{n!}{r!s!}$. Hilton and Pedersen have found that this viewpoint, and notation, is particularly useful in proving known theorems, and discovering new theorems, about binomial coefficients in Pascal's Triangle (see [4 - 6]).

Observe that $F(n,r,s) = \left(\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right)$ is an example of a special kind of function that we call separable. In particular, we say that a function F in three variables n, r, and s, is said to be a separable function if

$$F(n,r,s) = f(n)g(r)h(s).$$

Among the well-known separable functions (written in this expanded notation) are the Leibnitz Harmonic Coefficients and Gaussian Polynomials (or q-analogues).

We denote the Leibnitz Harmonic Coefficients by $\left[\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right]$ where $\left[\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right] = \frac{1}{(n+1)\left(\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right)}$, $0 \leq r, s \leq n$, and $\left[\begin{smallmatrix} & n & \\ 0 & & n \end{smallmatrix}\right] = \frac{1}{n+1}$.

The first seven rows of the Pascalian Harmonic triangle look like this:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & \frac{1}{2} & & \frac{1}{2} & & & & & \\
 & & & & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & & & & \\
 & & & \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{12} & & & \\
 & & \frac{1}{5} & & \frac{1}{20} & & \frac{1}{30} & & \frac{1}{20} & & \frac{1}{5} & & \\
 & \frac{1}{6} & & \frac{1}{30} & & \frac{1}{60} & & \frac{1}{60} & & \frac{1}{30} & & \frac{1}{6} & \\
\frac{1}{7} & & \frac{1}{42} & & \frac{1}{105} & & \frac{1}{140} & & \frac{1}{105} & & \frac{1}{42} & & \frac{1}{7}
\end{array} \tag{3}$$

You might like to check the Star of David theorem in this array when the center of the hexagon is at a general location, e.g. at $\frac{1}{12}$, $\frac{1}{20}$, $\frac{1}{30}$, or $\frac{1}{60}$. It is, of course, trivially true when the center of the pattern falls onto the vertical mirror-axis of symmetry.

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†Since $\left(\begin{smallmatrix} & n & \\ r & & s \end{smallmatrix}\right) = 0$ whenever r, s$> n$, the theorems are still true if the pattern extends beyond the non-zero entries of Pascal's Triangle.

We denote the Gaussian Polynomials (or q-analogues) by $\left\{ {n \atop r \quad s} \right\}$, where

$$\left\{ {n \atop r \quad s} \right\} = \frac{\prod_{i=0}^{n-1}(q^{n-1}-1)}{\prod_{i=0}^{r-1}(q^{r-i}-1)\prod_{i=0}^{s-1}(q^{s-1}-1)},$$

$0 \le r, s \le n$, $\left\{ {n \atop 0 \quad n} \right\} = 1$. When $q = 1$ the q-analogue becomes, not surprisingly, just the binomial coefficient. The first four rows of the Pascalian q-analogue triangle look like this:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & q+1 & & 1 & & \\
 & 1 & & q^2+q+1 & & q^2+q+1 & & 1 & \\
1 & & q^3+q^2+q+1 & & (q^2+1)(q^2+q+1) & & q^3+q^2+q+1 & & 1
\end{array} \tag{4}$$

The center term of the last row shown is, of course, $q^4 + q^3 + 2q^2 + q + 1$.

Using this notation, the binomial coefficients, $\binom{n}{r \quad s}$, naturally arrange themselves in a triangular array that looks like this

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0\ 0} & & & & \\
 & & & \binom{1}{0\ 1} & & \binom{1}{1\ 0} & & & \\
 & & \binom{2}{0\ 2} & & \binom{2}{1\ 1} & & \binom{2}{2\ 0} & & \\
 & \binom{3}{0\ 3} & & \binom{3}{1\ 2} & & \binom{3}{2\ 1} & & \binom{3}{3\ 0} & \\
\binom{4}{0\ 4} & & \binom{4}{1\ 3} & & \binom{4}{2\ 2} & & \binom{4}{3\ 1} & & \binom{4}{4\ 0}
\end{array} \tag{5}$$

Notice that the values of n, r, s are constant in the corresponding horizontal, upward slanting, and downward slanting directions. Note, too, that the values of n increase as you take successive horizontal rows lower down the triangle, the values of r increase as you take successive upward slanting rows further to the right in the triangle, and the values of s increase as you take successive downward slanting rows further to the left in the triangle. The following triangular figure in (6) illustrates the directions within Pascal's Triangle along which n, r, and s are constant.

$$\begin{array}{ccc}
\text{direction along which} \rightarrow & \diagup\ \diagdown & \leftarrow \text{direction along which} \\
r \text{ is constant} & \diagup \quad\ \ \diagdown & s \text{ is constant} \\
 & -\ -\ -\ - & \\
 & \uparrow & \\
 & \text{direction along which} & \\
 & n \text{ is constant} &
\end{array} \tag{6}$$

3. Some Known Results

The $\binom{n}{r \quad s}$ notation has an immediate payoff. In the diagrams that follow we will always indicate $\binom{n}{r \quad s}$ by $\bigstar$. Using this notation, and assuming, as before, that all entries shown are lattice points of a regular triangular grid that constitutes the entries of the Pascal Triangle, we can readily prove the

Star of David Theorem. *The product of the numbers located at the* $\clubsuit$ *vertices is the same as the numbers located at the* $\diamondsuit$ *vertices in array (7), which is a modified version of array* (1) *where we eliminate the dots outside the* 6*-gon and replace the center dot with* $\bigstar = \binom{\;\,n}{r\;\;s}$.

$$\begin{array}{ccccc} & \clubsuit & & \diamondsuit & \\ \diamondsuit & & \bigstar & & \clubsuit \\ & \clubsuit & & \diamondsuit & \end{array} \tag{7}$$

Proof. Since we know that $\bigstar = \binom{\;\,n}{r\;\;s}$ and that the other six symbols are next to $\bigstar$ each of the six binomial coefficients in (7) may be readily expressed as shown in display (8).

$$\begin{array}{ccccc} & \binom{\;\;n-1}{r-1\;\;s} & & \binom{\;\;n-1}{r\;\;s-1} & \\ \binom{\;\;\;n}{r-1\;\;s+1} & & \bigstar & & \binom{\;\;\;n}{r+1\;\;s-1} \\ & \binom{\;\;n+1}{r\;\;s+1} & & \binom{\;\;n+1}{r+1\;\;s} & \end{array} \tag{8}$$

Remembering that $\binom{\;\,n}{r\;\;s} = \frac{n!}{r!s!}$, look at the three horizontal directions passing through the hexagon in (8) to see that the product of the $\clubsuit$ and the $\diamondsuit$ coefficients would each include, in the numerator, $[(n-1)!]^2$, $[n!]^2$, and $[(n+1)!]^2$. Likewise, look at the three upward sloping directions in (8) (where the value of r is fixed) to see that the product of the $\clubsuit$ and $\diamondsuit$ coefficients would include, in the denominator, $[(r-1)!]^2$, $[r!]^2$, and $[(r+1)!]^2$. Finally, look at the three downward sloping directions in (8) (where the value of s is fixed) to see that the product of the $\clubsuit$ and $\diamondsuit$ coefficients would include, in the denominator, $[(s-1)!]^2$, $[s!]^2$, and $[(s-1)!]^2$. Thus we have

$$\prod \clubsuit = \frac{(n-1)!n!(n+1)!}{(r-1)!(r+1)!r!s!(s-1)!(s+1)!}$$

and that

$$\prod \diamondsuit = \frac{(n-1)!(n+1)!n!}{r!(r+1)!(r-1)!(s-1)!s!(s+1)!},$$

from which it follows that $\prod \clubsuit = \prod \diamondsuit$. ∎

An immediate consequence of this theorem is the following, which is actually listed as a theorem in the Hoggatt-Hansell paper [9].

Corollary to the Star of David Theorem. *Given the array in* (7), *with* $\binom{\;\,n}{r\;\;s}$, *it follows that* $\prod \clubsuit \prod \diamondsuit$ *is a perfect square.*

Results of this type are called perfect square patterns (PSP) and they are, in fact, of much interest to the authors of papers [1], [8], [9], [11–13], and [16].

In [12], Long repeatedly used a single result that he called "Lemma 1", as a basis for all of his proofs. We repeat this lemma here in our notation, with one added feature; that is, we replace Long's letters at the vertices of the two parallelograms with our symbols $\clubsuit$, $\diamondsuit$, $\heartsuit$, and $\spadesuit$, so that we may refer back to array (9) later and give our purely geometric proof that Long's Lemma 1 is true — and more!

Long's Lemma. [12] *In array* (9) *think of the parallelogram on the left having vertices marked by $\clubsuit$ and $\diamondsuit$, and the parallelogram on the right having vertices marked by $\heartsuit$ and $\spadesuit$. The product of the binomial coefficients at the "red" vertices is equal to that of the "black" vertices.*

$$
\begin{array}{ccccccccccccccccc}
\bullet & & \bullet & & \clubsuit & & \bullet & & \bullet & & \bullet & & \heartsuit & & \bullet & & \bullet \\
& \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
\bullet & & \bullet & & \bullet & & \diamondsuit & & \bullet & & \spadesuit & & \bullet & & \bullet & & \bullet \\
& \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
\diamondsuit & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \spadesuit \\
& \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
\bullet & & \clubsuit & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \heartsuit & & \bullet
\end{array}
\tag{9}
$$

Long and Hoggatt use Long's Lemma 1 to obtain a convex pear-shaped 8-gon with a vertical axis of symmetry (Figure 7 of [12]) for which the product of the 8 lattice points is a perfect square, and such that if the lattice points are labeled consecutively a_1, a_2, …, a_8, then the product of the even numbered vertices is equal to the product of the odd numbered vertices. We are looking for octagons with both a vertical and a horizontal axis of symmetry. In the Long-Ando paper [13]the main result concerns eighteen binomial coefficients lying on a convex hexagon such that, if numbered consecutively, say, clockwise as a_1, a_2, …, a_{18}, the product of the odd-numbered lattice points equals the product of the even-numbered lattice points. The geometric proof we will shortly describe will allow the reader to prove many of the theorems in the papers referred to at the end of this article, without writing out a single binomial coefficient — all you will need to do is to look at the figures and draw a few straight lines.

Long described a particular, but not special, case of the Generalized Star of David Theorem in [12]using his Lemma 1. Hilton and Pedersen discovered a class of Generalized Star of David Theorems, for which Long's is one case, in [4]and [5], by an entirely different route and described our general figure as arising from truncating an equilateral triangle to produce a semi-regular hexagon, where the size of the truncation may vary — but the same size triangle must be truncated from each vertex. The figure for this particular Star of David Theorem is shown in array (10), using ♡ and ♠ as markers for the vertices of the bounding hexagon.

$$\begin{array}{c}
\heartsuit \quad \bullet \quad \spadesuit \\
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \quad \bullet \quad \bigstar \quad \bullet \quad \bullet \quad \bullet \\
\spadesuit \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \heartsuit \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\heartsuit \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \spadesuit
\end{array} \tag{10}$$

A Generalized Star of David Theorem. *Given the array in* (7), *with* $\bigstar = \binom{n}{r\ s}$, *it follows that* $\prod \heartsuit = \prod \spadesuit$.

Proof. (The hard way!) Write out each of the binomial coefficients represented — for example the heart on the top row is $\binom{n-1}{r-3\ s-1}$, etc, to see that taking the hearts in clockwise order we have

$$\begin{aligned}
\prod \heartsuit &= \binom{n-4}{r-3\ \ s-1}\binom{n+1}{r+4\ \ s-3}\binom{n+3}{r-1\ \ s+4} \\
&= \frac{(n-4)!(n+1)!(n+3)!}{(r-3)!(s-1)!(r+4)!(s-3)!(r-1)!(s+4)!};
\end{aligned}$$

similarly, beginning with the top spade and continuing in clockwise order we have

$$\begin{aligned}
\prod \spadesuit &= \binom{n-4}{r-1\ \ s-3}\binom{n+3}{r+4\ \ s-1}\binom{n+1}{r-3\ \ s+4} \\
&= \frac{(n-4)!(n+3)!(n+1)!}{(r-1)!(s-3)!(r+4)!(s-1)!(r-3)!(s+4)!}.
\end{aligned}$$

Comparing the factors on the right-hand side of the last two expressions, we have $\prod \heartsuit = \prod \spadesuit$. ∎

You will suspect from our introduction that it is possible to obtain a much easier proof that works not only for this theorem, but for any theorem similar to it involving an even number of vertices. Having seen the harder proof above, and having played the geometric game given in the introduction, you will, no doubt, be able to see why this geometric proof works, and why the geometric proof of this theorem is just a particular case of a more general method.

A Better Proof of a Generalized Star of David Theorem. (Let geometry do the work!) First, in array (10), ignoring the ★, observe that along each horizontal line (where n is fixed and either a ♡ or a ♠ occurs) there are the same number of ♡ and ♠ (exactly one of each in this case). Second, observe that along each upward sloping dashed line (where r is fixed and either a $\heartsuit$ or a $\spadesuit$ occurs) there are the same number of $\heartsuit$ and $\spadesuit$ (again, one of each in this case). Third, observe that along each downward sloping dashed line (where s is fixed and either a $\heartsuit$ or a $\spadesuit$ occurs) there are the same number of $\heartsuit$ and $\spadesuit$ (yet again, one of each in this case). Finally, observe that if you had drawn straight dashed lines in each direction then every $\heartsuit$ and every $\spadesuit$ in the array would have a horizontal, an upward sloping, and a downward sloping dashed line passing through it that contains its counterpart and the figure would appear as shown in array (11). We have drawn the lines in array (11) slightly offset, so that they don't obscure the suit symbols, which creates an equilateral triangle bounded by dashed lines about each $\heartsuit$ and each $\spadesuit$.

$$\begin{array}{ccccccccccccccccc}
 & & & & & \diagup & \heartsuit & \diagdown & \bullet & \diagup & \spadesuit & \diagdown & & & & & \\
 & & & & \diagup & \bullet & - & \bullet & \times & \bullet & - & \bullet & \diagdown & & & & \\
 & & & \diagup & \bullet & & \bullet & \diagup & \bullet & \diagdown & \bullet & & \bullet & \diagdown & & & \\
 & & \diagup & \bullet & & \bullet & \diagup & \bullet & & \bullet & \diagdown & \bullet & & \bullet & \diagdown & & \\
 & \diagup & \bullet & & \bullet & \diagup & \bullet & & \bigstar & & \bullet & \diagdown & \bullet & & \bullet & \diagdown & \\
\diagup & \spadesuit & \diagdown & \bullet & \diagup & \bullet & & \bullet & & \bullet & & \bullet & \diagdown & \bullet & \diagup & \heartsuit & \diagdown \\
- & - & \bullet & \times & \bullet & - & \bullet & - & \bullet & - & \bullet & - & \bullet & \times & \bullet & - & - \\
 & & \diagup & \heartsuit & \diagdown & \bullet & & \bullet & & \bullet & & \bullet & \diagup & \spadesuit & \diagdown & & \\
 & \diagup & - & - & - & \diagdown & - & - & - & - & - & \diagup & - & - & - & \diagdown &
\end{array} \tag{11}$$

■

This proof obviously generalizes to situations where more than two lattice points lie on the lines where n, r, or s is fixed — and there is no need even to have $\bigstar$ in the diagram. In those cases there must always be an even number of entries along each of the lines in each of the three directions (and it need not be the same even number of entries for each line). Now, look again at Long's Lemma 1 and you will see that if you think of the actual colors of the four suits then you can use this technique to prove that in display (9)

$$\prod (\text{red vertices}) = \prod (\text{black vertices}),$$

or, more symbolically, that $\prod \clubsuit\spadesuit = \prod \diamondsuit\heartsuit$. If you prove this first, then Long's Lemma 1 is an immediate consequence.

As another example we state, in our notation, the main theorem of [13]as the

Long-Ando Theorem. [14] *Given the array in* (12) *it follows that* $\prod \clubsuit = \prod \diamondsuit$. (*We include* $\bigstar$ *because Long and Ando marked this location, but it has nothing to do with the proof.*)

$$\begin{array}{ccccccccccccccc}
 & \clubsuit & & \diamondsuit & & \clubsuit & & \diamondsuit & & \clubsuit & & \diamondsuit & & & \\
\diamondsuit & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \clubsuit & & \\
 & \clubsuit & & \bullet & & \bullet & & \bigstar & & \bullet & & \bullet & & \diamondsuit & \\
 & & \diamondsuit & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \clubsuit \\
 & & & \clubsuit & & \diamondsuit & & \clubsuit & & \diamondsuit & & \clubsuit & & \diamondsuit &
\end{array} \tag{12}$$

Proof. Simply draw the horizontal, upward sloping, and downward sloping grid lines in all directions where they would pass through any ♣ or ♢. Then note that along each line you always have the same number of ♣ as ♢. Equivalently, every ♣ and every ♢ has three lines drawn through it and on each of those three lines we find the same number of black and red entries. When these lines are drawn slightly offset, as in array (11), each ♣ and each ♢ will be enclosed by an equilateral triangle. ∎

Of course an immediate consequence of the Long-Ando Theorem is that the product $\prod \clubsuit \prod \diamondsuit$ is a perfect square, so we may say that array (12) is a (PSP).

Owing to the underlying geometry, an additional feature of this type of theorem is that any array involved may be rotated $\pm 60^\circ$, about any lattice point, without changing the truth of the theorem.

It is interesting that Usiskin (see [16]), in fact, saw the principle involved in our geometric proof when he wrote about perfect square patterns (PSP), stating "If a pattern contains an even number of elements on every horizontal line and major diagonal, then the pattern is a PSP". He gives many such examples, with varying types of symmetry. Usiskin doesn't limit himself to convex polygons, or even to polygons. His patterns are quite ingenious, sometimes involving the union of two or more disjoint PSP's and, by this device, he even manages to produce perfect cube patterns. You may wish to try our geometric proof on the various patterns of [16].

Before we close this section we should ask you to observe that, in the Star of David Theorems we have shown that the greatest common divisor of the ♣ and the greatest common divisor of the ♢ were, in all cases, equal to each other. Or, more mathematically, $\gcd(\clubsuit) = \gcd(\diamondsuit)$. Exactly when this property holds was of significant interest to the authors of references [2], [14], and [15].

4. Proving 2*n*-Star Theorems

To describe theorems analogous to the Star of David Theorems for $2n$ vertices, with $n \geq 3$, we will call a *convex* polygon created by $2n$ vertices that lie on the triangular lattice points a complete **2*n*-gon**, if

(a) the $2n$ vertices can be partitioned into two disjoint sets (say ♡ and ♠), each having n vertices.

(b) the two polygons obtained by connecting the n vertices of like types are congruent to each other.

We call the two n-gons, whose union forms the convex $2n$-gon, complementary ***n*-gons**. In this notation, the Star of David Theorem involves a complete 6-gon that is decomposed into two complementary 3-gons.

If a complete $2n$-gon can be decomposed into two complementary n-gons such that the product of the numbers located at the vertices of one of the n-gons is equal to the product of the numbers located at the vertices of its complementary

n-gon, then we call this a $\boldsymbol{2n}$-Star Theorem, or $\boldsymbol{2n}$-Star Pattern, (when no more picturesque name suggests itself).

In this notation the Star of David Theorem would be called a 6-Star Theorem.

As you can well believe from the "game" we presented in the introduction, and the discussion of the geometric proof above, the method of proof for $2n$-Star Theorems is to show that

(i) the given $2n$-gon contains a pair of elements on every line passing through vertices of the figure in the directions for which n, r, and s are constant, or equivalently, that

(ii) every entry must have a horizontal, upward sloping, and downward sloping grid line through it containing its counterpart.

Of course, as we have seen, each theorem of this type produces, as a trivial consequence, a PSP pattern — but as Usiskin showed us in [16], there are other ways to get PSP patterns.

Gould referred, in [1], to the theorem shown in arrays (1) and (7) as the Star of David Theorems and focused his attention on looking for "equal products of generalized coefficients" (always binomial coefficients). His approach was a clever combination of choosing values for n (usually an arithmetic sequence) and then finding the appropriate values for r in order to make the final collection of coefficients a PSP. Gould found, among other things, equal products of binomial coefficients where the total number of coefficients was 4, 6, 8, and 10. The array in (13), which also appears in [16], is an example of Gould's arrangement for 8 coefficients that form a convex 8-gon. Thus we have an 8-Star Theorem that we call

Gould's First Theorem. [1] *Given the array in* (13)*, it follows that* $\prod \clubsuit = \prod \diamondsuit$*, regardless of where this pattern is placed on Pascal's Triangle.*

$$\begin{array}{cccccccc}
 & & & & \clubsuit & & \diamondsuit & \\
 & \diamondsuit & & \bullet & & \bullet & & \clubsuit \\
\clubsuit & & \bullet & & \bullet & & \diamondsuit & \\
 & \diamondsuit & & \clubsuit & & & &
\end{array} \tag{13}$$

We feel confident the reader can carry out the geometric proof, by checking in array (13), either that (a) there are four lines in each of three constant directions that contain one $\clubsuit$ and one $\diamondsuit$, or that (b) every $\clubsuit$ (or $\diamondsuit$) entry has a counterpart $\diamondsuit$ (or $\clubsuit$) in each of the three directions where n, r, or s is constant.

What is satisfying, to us, is that the vertices created by the $\clubsuit$ and $\diamondsuit$ form a semi-regular octagon with two axes of symmetry — but note that this octagon is not centered about any single lattice point $\bigstar = \binom{n}{r\ s}$.

Gould carried this idea further and obtained the following theorem (which also appears in [16]) involving 10 lattice points.

Gould's Second Theorem. [1] *Given the array in* (14)*, it follows that* $\prod \clubsuit = \prod \diamondsuit$*,* *regardless of where this pattern is placed on Pascal's Triangle.*

$$\begin{array}{ccccccccc}
 & & \clubsuit & & \diamondsuit & & & & \\
 & \bullet & & \bullet & & \clubsuit & & \diamondsuit & \\
\diamondsuit & & \bullet & & \bullet & & \bullet & & \clubsuit \\
 & \clubsuit & & \diamondsuit & & \bullet & & \bullet & \\
 & & & & \clubsuit & & \diamondsuit & &
\end{array} \tag{14}$$

The geometric proof causes no difficulty.

Satisfying as this result is, we notice that the bounding figure of the entire array in (14) is not a convex 10-gon — and that is why we didn't refer to it as a 10-Star Theorem. Furthermore, the bounding figure of the $\clubsuit$ is not even a convex 5-gon — nor, of course, is the bounding figure of the $\diamondsuit$ — since, in both cases, three of the lattice points lie on a straight line (that, incidentally, is not a line parallel to the r or s direction). We will shortly show you a 10-Star Theorem that does not have these defects.

5. New Results

First we present a few of our 8-Star Patterns which Hilton and Pedersen wrote about in [6]. Laura Brown, a student of Pedersen, suggested that these patterns should be called Stop-Sign Patterns, because, for large symmetric ones, the bounding 8-gons "look like Stop-Signs."

Stop-Sign Pattern 1. *Given the array in* (15), *it follows that* $\prod \heartsuit = \prod \spadesuit$, *regardless of where this pattern is placed on Pascal's Triangle.*

$$\begin{array}{ccccccc}
 & & \heartsuit & & \spadesuit & & \\
 & \bullet & & \bullet & & \bullet & \\
\spadesuit & & \bullet & & \bullet & & \heartsuit \\
 & \bullet & & \bullet & & \bullet & \\
\heartsuit & & \bullet & & \bullet & & \spadesuit \\
 & \bullet & & \bullet & & \bullet & \\
 & & \spadesuit & & \heartsuit & &
\end{array} \tag{15}$$

Stop-Sign Pattern 2. *Given the array in* (16), *it follows that* $\prod \heartsuit = \prod \spadesuit$, *regardless of where this pattern is placed on Pascal's Triangle.*

$$\begin{array}{ccccccccccc}
 & \heartsuit & & \bullet & & \spadesuit & & & & & \\
\spadesuit & & \bullet & & \bullet & & \bullet & & \heartsuit & & \\
 & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
 & & \heartsuit & & \bullet & & \bullet & & \bullet & & \spadesuit \\
 & & & & & \spadesuit & & \bullet & & \heartsuit &
\end{array} \tag{16}$$

Stop-Sign Pattern 3. *Given the array in* (17), *it follows that* $\prod \heartsuit = \prod \spadesuit$, *regardless of where this pattern is placed on Pascal's Triangle.*

$$\begin{array}{ccccccccccc}
 & & & & \heartsuit & & \spadesuit & & & & \\
 & & & \bullet & & \bullet & & \bullet & & & \\
 & & \bullet & & \bullet & & \bullet & & \bullet & & \\
 & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
\spadesuit & & \bullet & & \bullet & & \bullet & & \bullet & & \heartsuit \\
 & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
\heartsuit & & \bullet & & \bullet & & \bullet & & \bullet & & \spadesuit \\
 & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
 & & \bullet & & \bullet & & \bullet & & \bullet & & \\
 & & & \bullet & & \bullet & & \bullet & & & \\
 & & & & \spadesuit & & \heartsuit & & & &
\end{array} \tag{17}$$

Array (18) shows a 10-Star Pattern that was discovered by Séquin within minutes of the challenge given at the end of a talk that Pedersen gave at the eighth Gathering for Gardner Conference (G4G8) which was held in Atlanta, Georgia, in March, 2008. It was this discovery, and subsequent discussions between the authors that led to the current paper.

$$\begin{array}{ccccccccccccccccc}
 & & & & & \bullet & & \heartsuit & & \spadesuit & & \bullet & & & & & \\
 & & & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & & & \\
 & & & \bullet & & \spadesuit & & \bullet & & \bullet & & \heartsuit & & \bullet & & & \\
 & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \\
 & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
\bullet & & \bullet & & \heartsuit & & \bullet & & \bullet & & \bullet & & \spadesuit & & \bullet & & \bullet \\
 & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & \\
 & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \\
 & & & \bullet & & \spadesuit & & \bullet & & \bullet & & \heartsuit & & \bullet & & & \\
 & & & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet & & & & \\
 & & & & & \bullet & & \heartsuit & & \spadesuit & & \bullet & & & & &
\end{array} \tag{18}$$

10-Star Pattern. *Given the array in* (18), *it follows that* $\prod \heartsuit = \prod \spadesuit$, *regardless of where this pattern is placed on Pascal's Triangle.*

6. Complete $2n$-Star Patterns with Constraints

The previous sections have been a gentle introduction to a much more general geometrical placement game/puzzle/challenge, which we now describe.

How to place two sets of markers, i.e. n grey (round) ones and n black (square) ones, onto the triangle (or 3-line) grid of a 3-component separable function, so that the product of the terms at the grey locations equals the product of the terms at the black locations.

As we have shown in previous sections, this challenge can be seen as a geometrical placement problem:

Place the two sets of n markers onto the triangle grid in such a way that

(a) every grid line contains an equal set of round and square markers, and
(b) every marker sits on three different grid lines, which each contain a complementary marker (of the other type).
(c) the n-gon bounded by the round markers is congruent to the n-gon bounded by the square markers. These two polygons are called complementary ***n*-gons**.

When the above conditions are met we will call the result a $2n$-Star pattern. The reader may guess by now – correctly – that complete $2n$-Star patterns are possible for arbitrarily large n values, and that the number of possible patterns satisfying these general constraints may be infinite. Thus the question arises how to find appropriate placements of the $2n$ markers. Trial and error is a method that works quite well for n values up to 20 or so. Using an interactive drawing program that makes it easy to replicate lines and shift them parallel to themselves so as to pass through an arbitrary location, one can usually find a pattern that meets all constraints in less than an hour.

Thus, in order to make the puzzle more interesting (challenging!) it seems appropriate to introduce the following four extra constraints – which can be viewed as design goals to produce particularly attractive star patterns:

(i) Maximize the symmetry of the patterns. (In the previous sections we have typically displayed patterns that had horizontal and vertical bilateral symmetry axes.)
(ii) Restrict the patterns to true convex polygons in which all the markers form the vertices of a $2n$-gon; i.e., no two markers of the same color lie on the same grid line.
(iii) Try to keep the pattern as compact as possible, i.e., to minimize
    (a) the area of the $2n$-gon (measured in the number of triangles), or
    (b) its circumference, or
    (c) the size of its axis-aligned bounding box, i.e.,(max-height)$\times$(max-width).
(iv) Emphasize an "even distribution;" i.e., markers should lie approximately on an ellipse and should all have about the same distances from their nearest neighbors.

As an example, the $2n$-Star Patterns shown in Figure 1 were obtained by applying design constraints (1), (2), and to some degree also (3) and (4). Note that symmetry (ignoring the type, or shape, of the vertex markers) has been prioritized in these patterns: they all have at least two mutually perpendicular bilateral symmetry axes. For the cases where n is divisible by 3, the patterns show 6 axes of symmetry, 30° apart.

We now describe some of the heuristics that we have used to construct these patterns. We look at the case of $n = 7$ in Figure 2, where we show several different solutions emphasizing different design goals. Say we want to construct

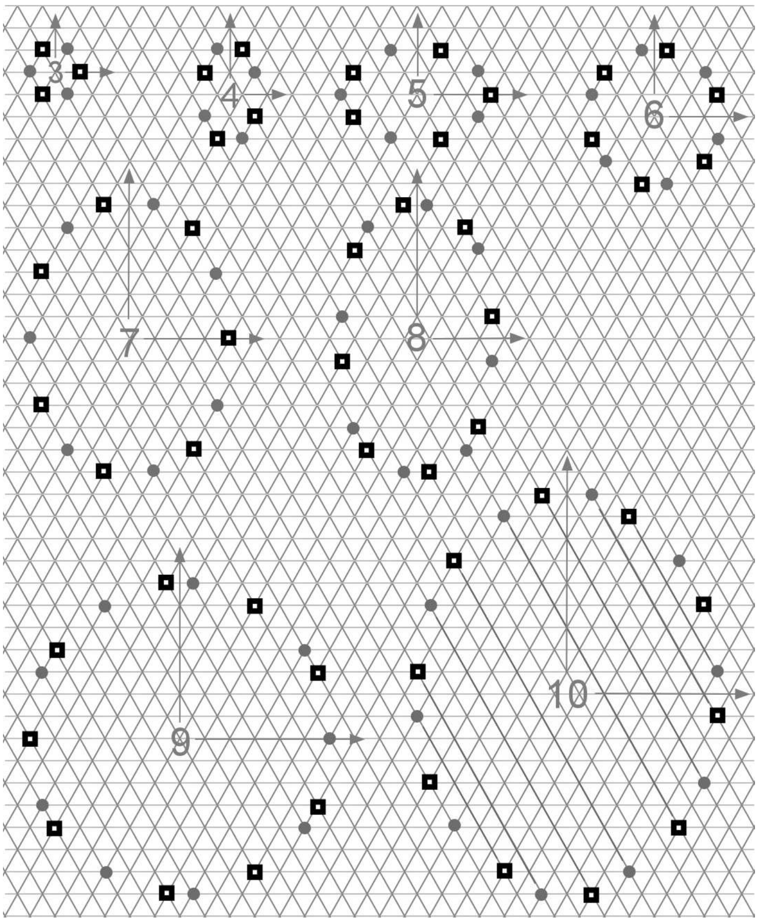

Fig. 1. $2n$-Star Patterns, $3 \leqq n \leqq 10$ that satisfies constraints (i) and (ii) of this section.

a highly symmetrical and evenly spaced pattern for the case of $n = 7$ (as shown in Figure 2a). We may start with the two markers #1 and #2 at the top and space them two units apart – this is a rough guess as to what an average spacing might be on a circle or ellipse large enough to accommodate all 14 markers. On average every consecutive side of the 14-gon would rotate through $360/14 \approx 25.714^\circ$ (clockwise) at each vertex as we walk around the 14-gon in a clockwise manner.

Thus, starting from the first two markers with equal y-values, for our next marker location (#3) we look in a downward direction of about 26° from the second marker for a close-by lattice point, and we find such a point two triangle heights away. Now for marker #4 we need to look along a slope angle about twice as large. We actually choose a slope of 60° and thereby automatically get a pair of markers lined up on one of the slanted grid lines. We pick a point two triangle sides away, again to satisfy our evenness goal. For the next location (#5) we look in a direction of about 77°, and also try to find a location that falls on the slanted grid line that passes through marker #2 — and we find the black marker on the x-axis. Because of the desired symmetry, all other markers are now determined. Now we have to check along the slanted grid lines to make sure that each marker finds its complementary companion on such a grid line. If such complementary pairs do not yet line up, one or the other, or both markers, will have to be moved slightly to obtain all the required alignments.

We should also try to add markers by starting with the extreme point on the x-axis (the last one placed above; #1 in Figure 2b) and work our way counter-clockwise to the top of the figure. In this case we can obtain a significantly smaller figure – particularly, if we also drop the evenness constraint and start with the smallest possible non-vertical step from marker #1 to marker #2. The result, Figure 2b, encloses only 106 grid triangles, as compared with the 152 triangles circumnavigated by the polygon in Figure 7a.

In general, the required pairing of markers leads to a slicing of the $2n$-gon in all directions parallel to a given grid line. For larger n-values it is often productive to start in the first quadrant, picking a slanted grid line with a slope of 60° to place a first pair of markers, and then to continue creating matched pairs on parallel grid lines, working across ever longer spans until a complete quarter has been built. During this construction, we should also pencil in the markers symmetrically reflected across the x- and the y-axes, and check that we get the required pairings in the other grid line directions. The spacings between the markers need to be adjusted as necessary. With only three grid line directions, this task is not too daunting.

Cases c and d of Figure 2 show other patterns with somewhat different design constraints. Figure 2c shows a 14-gon closely related to Figure 2a, but with a smaller area; it was obtained by eliminating the up-down symmetry. Figure 2d shows that we can obtain a polygon with much less area if we eliminate the non-collinearity requirement; now the area is only 67 triangles, but the bounding polygon now has only six distinct corner vertices!

Once we abandon the strict convexity constraint, we can, of course, group many more markers in much denser symmetrical clusters. Figure 3 shows some example patterns.

7. Generalization to Higher-Order Separable Functions

The key to forming these complementary n-gon patterns with equal products of the values at their marker locations is a two-dimensional array of numbers based on a separable function with three product terms, where each product term remains constant along a set of parallel grid lines with three different slopes. Thus it seemed worthwhile to see whether one can construct patterns on number arrays based on separable functions with four or more terms – and indeed we

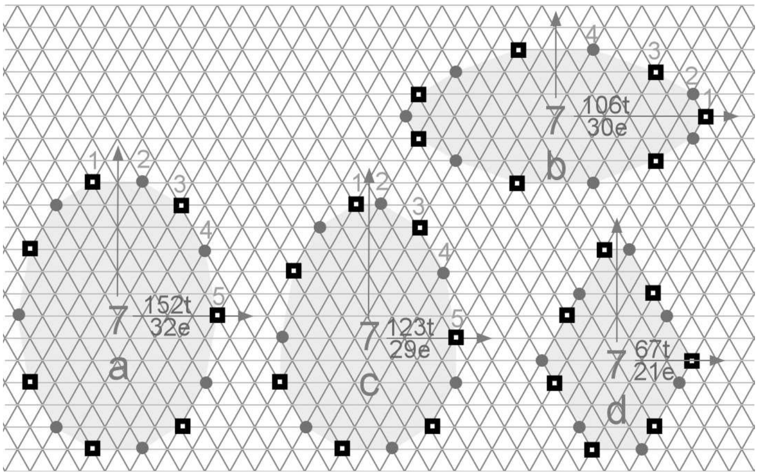

Fig. 2. A variety of 14-Star Patterns.

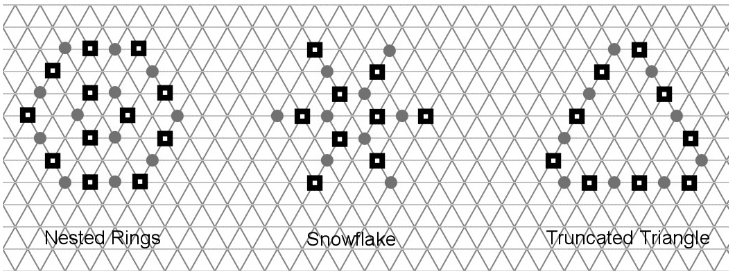

Fig. 3. Complete $2n$-Cluster Patterns.

can!

To illustrate the principle, we construct a simple separable function F_4 with four product terms as follows: $F_4\,(x, y, (x+y), (x-y)) = xy(x+y)(x-y)$.

The individual product terms remain constant on vertical, horizontal, and sloped gridlines with slopes of -1 and $+1$, respectively. Thus, in order to construct patterns analogous to what we have discussed so far, we need to create two complementary marker sets where every one of the four grid lines with the above mentioned slopes passing through one of the markers also carries one of the complementary markers. An example with two sets of four markers forming a convex octagon is highlighted in the spreadsheet, shown in Figure 4, based on the function F_4 above. Other examples of such $2n$-gon patterns on a symmetrical 4-line (or quad) grid are shown in Figure 5.

Setting these separable functions up on a spreadsheet is an easy and quick demonstration of the movability of the $2n$-gon patterns. At the bottom of the

F4symm = X * Y * (X+Y) * (X-Y)

	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
8	0	840	1344	1560	1536	1320	960	504	0	-504	-960	-1320	-1536	-1560	-1344	-840	0
7	-840	0	546	840	924	840	630	336	0	-336	-630	-840	-924	-840	-546	0	840
6	-1344	-546	0	330	480	486	384	210	0	-210	-384	-486	-480	-330	0	546	1344
5	-1560	-840	-330	0	180	240	210	120	0	-120	-210	-240	-180	0	330	840	1560
4	-1536	-924	-480	-180	0	84	96	60	0	-60	-96	-84	0	180	480	924	1536
3	-1320	-840	-486	-240	-84	0	30	24	0	-24	-30	0	84	240	486	840	1320
2	-960	-630	-384	-210	-96	-30	0	6	0	-6	0	30	96	210	384	630	960
1	-504	-336	-210	-120	-60	-24	-6	0	0	0	6	24	60	120	210	336	504
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	504	336	210	120	60	24	6	0	0	0	-6	-24	-60	-120	-210	-336	-504
-2	960	630	384	210	96	30	0	-6	0	6	0	-30	-96	-210	-384	-630	-960
-3	1320	840	486	240	84	0	-30	-24	0	24	30	0	-84	-240	-486	-840	-1320
-4	1536	924	480	180	0	-84	-96	-60	0	60	96	84	0	-180	-480	-924	-1536
-5	1560	840	330	0	-180	-240	-210	-120	0	120	210	240	180	0	-330	-840	-1560
-6	1344	546	0	-330	-480	-486	-384	-210	0	210	384	486	480	330	0	-546	-1344
-7	840	0	-546	-840	-924	-840	-630	-336	0	336	630	840	924	840	546	0	-840
-8	0	-840	-1344	-1560	-1536	-1320	-960	-504	0	504	960	1320	1536	1560	1344	840	0

1219276800 equals 1219276800

Fig. 4 Spreadsheet for a convex 8-Star Pattern, based on F_4, where $(-30)(-840)(504)(96) = (-24)(-630)(960)(84) = 1219276800$

array is an area where the product functions for the round and square markers are calculated (separated by the field with the word "equals"). If these three horizontally adjacent cells are copied to a different location, then the corresponding functions will pick up their input from correspondingly shifted locations. One can then observe that the two products remain indeed equal no matter where the marker template is placed. The basic three-grid array based on the binomial function is shown in the spreadsheet of Figure 6.

This success emboldened us to tackle even higher-order separable functions. Figure 7 shows an example of an array generated by F_5, with five product terms, where $F_5 = y(x+y)(x+3y)(x-y)(x-3y)$.

Highlighted in the spreadsheet in Figure 7 are two sets of six marker locations that form a symmetrical 12-gon. (Note that the limited-precision arithmetic of this Excel spreadsheet does not compute products accurately.)

Correspondingly, Figure 8 then shows several patterns of $2n$-gons for $n = 6$ and for $n = 12$. They were all derived from the large convex 24-gon surrounding all other patterns by eliminating alternate pairs of markers on the perimeter and then moving the remaining pairs closer to the center of the figure. All $2n$-gons meet all the alignment conditions along all five grid line directions and thus provide the same products at all the round marker positions and at all the

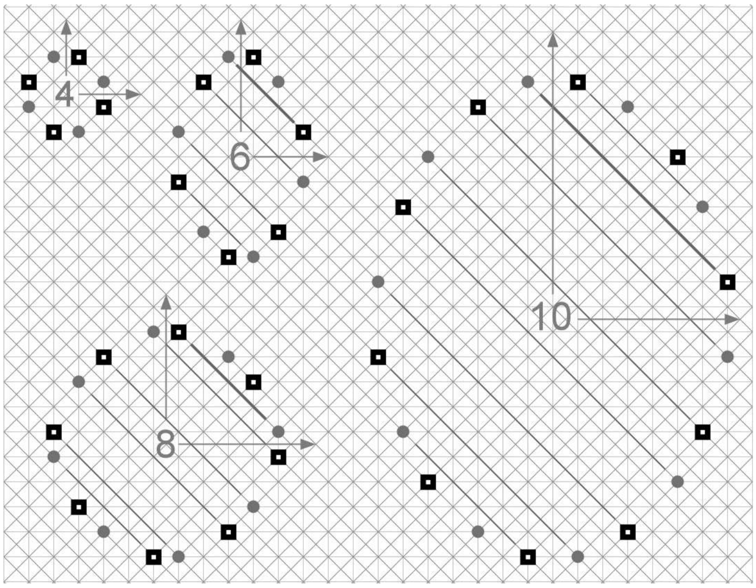

Fig. 5. Examples of 8-, 12-, 16-, and 20-Star Patterns on a 4-line grid.

square ones regardless of where the pattern is placed onto this array — hence they all produce complete $2n$-Star Patterns.

The search for this pattern made it quite clear that the trial-and-error method becomes harder and harder as the order of the separable function increases, and thus the number of constraints that have to be observed gets larger. Even for relatively small patterns it took us more than an hour to move all the markers around, trying to meet all five alignment conditions for each one of them.

One might try to picture this task as playing with a complicated mechanical linkage in which all markers sit at the joints of several rods, which themselves are constrained to be parallel to certain directions. The x- and y-positions of the markers constitute variables, and the rods present constraints. It is clear that some "mechanisms" of this kind may be over-constrained, i.e., there are more constraints than degrees of freedom. For instance, we were not yet able to find a solution for creating the "simplest" case consisting of a convex 10-gon on the order-5 array. Conceptually it is clear what the figure might look like, if the slopes of all five functions would correspond exactly to 5 angular directions distributed completely uniformly in the plane, and if we were not constrained to a discrete grid, the solution would then be the decagon shown in Figure 9. However, the various irrational ratios (e.g., equal to the golden mean) cannot be realized with integer lattice steps on a planar grid.

To explore this issue further, we also generated a gridded array corresponding to a sixth order separable function. A conceptual 2×6-gon realizable in a

Binomial: (X+Y)! / X! / Y!

	0	1	2	3	4	5	6	7	8	9	10	11
11	1	12	78	364	1365	4368	12376	31824	75582	167960	352716	705432
10	1	11	66	286	1001	3003	8008	19448	43758	92378	184756	352716
9	1	10	55	220	715	2002	5005	11440	24310	48620	92378	167960
8	1	9	45	165	495	1287	3003	6435	12870	24310	43758	75582
7	1	8	36	120	330	792	1716	3432	6435	11440	19448	31824
6	1	7	28	84	210	462	924	1716	3003	5005	8008	12376
5	1	6	21	56	126	252	462	792	1287	2002	3003	4368
4	1	5	15	35	70	126	210	330	495	715	1001	1365
3	1	4	10	20	35	56	84	120	165	220	286	364
2	1	3	6	10	15	21	28	36	45	55	66	78
1	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	1

			3298014720	EQUALS	3298014720					
13860	EQUALS	13860	850269420	EQUALS	850269420	1.05229E+12	EQUALS	1.05229E+12		
6930	EQUALS	6930				1.25273E+11	EQUALS	1.25273E+11		
3150	EQUALS	3150	35675640	EQUALS	35675640	11903771880	EQUALS	11903771880		
1260	EQUALS	1260				850269420	EQUALS	850269420		
			582120	EQUALS	582120		128828700	EQUALS	128828700	
105	EQUALS	105					2882880	EQUALS	2882880	
15	EQUALS	15	1260	EQUALS	1260			0	EQUALS	0
0	EQUALS	0	0	EQUALS	0					

Fig. 6 "Pascal Triangle" printed in a rectangular array.

continuous space in which six linear product functions are superposed at angles of 30° increments is shown in Figure 10.

As a challenge, you may try to design some convex $2n$-Star Patterns on the gridded template shown in Figure 11 . This grid corresponds to the 6-order separable function: $F_{6a} = xy(2x+y)(2x-y)(x+2y)(x-2y)$.

Alternatively we can construct a different 6-order separable function which is derived from the five-line grid of Figure 7 by simply adding one more product term (corresponding to vertical grid lines): $F_{6b} = xy(x-y)(x-y)(x+3y)(x-3y)$.

The function F_{6b} allows us to copy some of the $n = 6$ and $n = 12$ patterns onto this six-line grid, since the original patterns already have vertical alignments, even though the 5-line grid did not have any grid lines in that direction. The new lattice of Figure 12 is obtained by adding the vertical grid lines to the five-line grid of Figure 8 grid and stretching it horizontally by a factor of 2.

8. Open Questions

There still remain many questions to be explored. For example:

(i) In the context of Section 7: What are the slope values that enable or prevent a solution? For instance, we have not yet been able to find a

F5a = Y * (X+3Y) * (X+Y) * (X-3Y) * (X-Y)

	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
8	0	63240	120960	171912	215040	249480	274560	289800	294912	289800	274560	249480	215040	171912	120960	63240	0
7	-39585	0	36855	69888	98175	120960	137655	147840	151263	147840	137655	120960	98175	69888	36855	0	-39585
6	-43680	-21450	0	19734	36960	51030	61440	67830	69984	67830	61440	51030	36960	19734	0	-21450	-43680
5	-31395	-21120	-10395	0	9405	17280	23205	26880	28125	26880	23205	17280	9405	0	-10395	-21120	-31395
4	-15360	-12540	-8640	-4284	0	3780	6720	8580	9216	8580	6720	3780	0	-4284	-8640	-12540	-15360
3	-2805	-3840	-3645	-2688	-1365	0	1155	1920	2187	1920	1155	0	-1365	-2688	-3645	-3840	-2805
2	3360	1170	0	-462	-480	-270	0	210	288	210	0	-270	-480	-462	0	1170	3360
1	3465	1920	945	384	105	0	-15	0	9	0	-15	0	105	384	945	1920	3465
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	-3465	-1920	-945	-384	-105	0	15	0	-9	0	15	0	-105	-384	-945	-1920	-3465
-2	-3360	-1170	0	462	480	270	0	-210	-288	-210	0	270	480	462	0	-1170	-3360
-3	2805	3840	3645	2688	1365	0	-1155	-1920	-2187	-1920	-1155	0	1365	2688	3645	3840	2805
-4	15360	12540	8640	4284	0	-3780	-6720	-8580	-9216	-8580	-6720	-3780	0	4284	8640	12540	15360
-5	31395	21120	10395	0	-9405	-17280	-23205	-26880	-28125	-26880	-23205	-17280	-9405	0	10395	21120	31395
-6	43680	21450	0	-19734	-36960	-51030	-61440	-67830	-69984	-67830	-61440	-51030	-36960	-19734	0	21450	43680
-7	39585	0	-36855	-69888	-98175	-120960	-137655	-147840	-151263	-147840	-137655	-120960	-98175	-69888	-36855	0	39585
-8	0	-63240	-120960	-171912	-215040	-249480	-274560	-289800	-294912	-289800	-274560	-249480	-215040	-171912	-120960	-63240	0

4237818177017110000000 equals 4237818177017110000000

Fig. 7. The array generated by F_5.

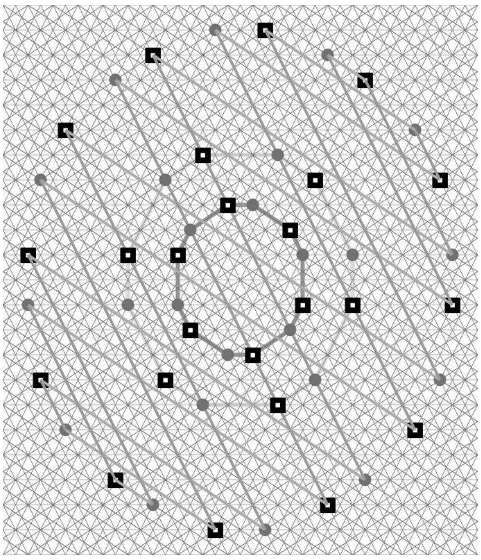

Fig. 8. $2n$-gon Star Patterns for $n = 6$ and 12 on a 5-line grid.

solution for $n = 6$ on the grid corresponding to the function F_{6a}, with symmetrical 2:1 and 1:2 slopes.

(ii) For which of the $2n$-Star Patterns (or theorems) is it the case that the

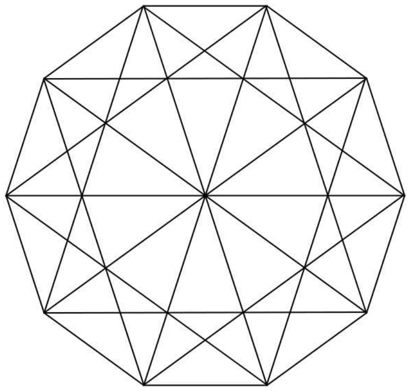

Fig. 9. Decagram

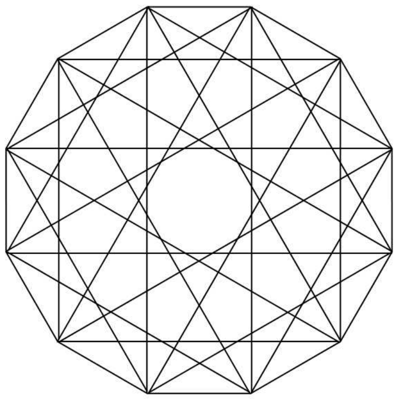

Fig. 10 Dodecagram

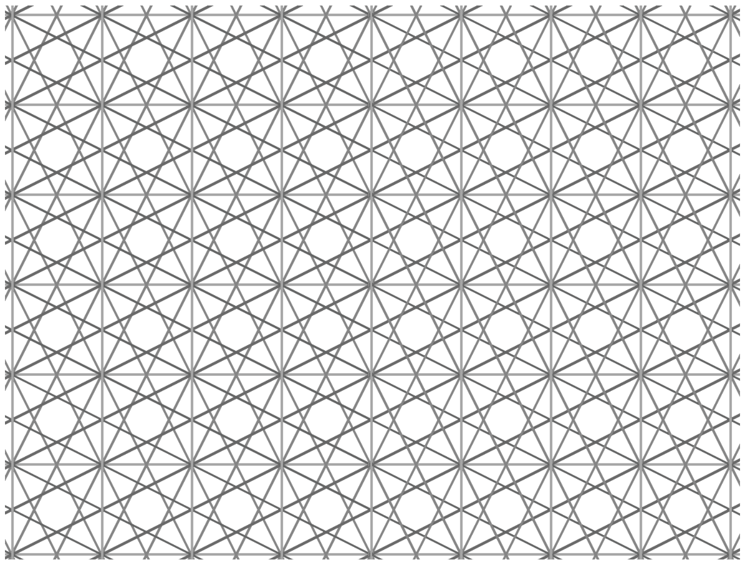

Fig. 11. Symmetrical 6-line grid.

gcd of the entries of one n-gon is equal to the gcd of the entries of its complementary.

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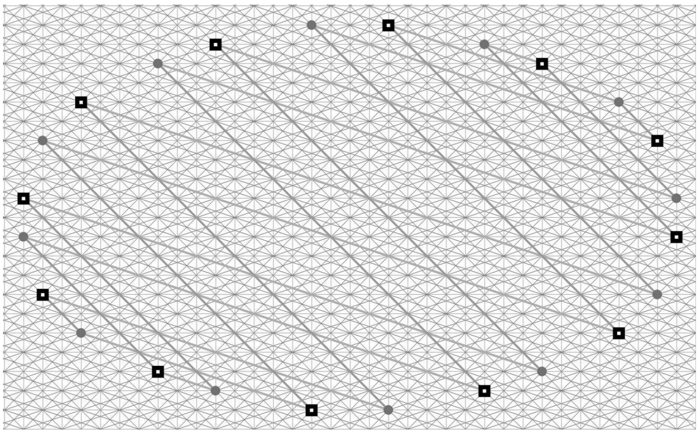

Fig. 12 A 24-Star Pattern on a 6-line grid.

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