Interactive Construction of 3D Mathematical Visualization Models

Carlo H. Séquin <sequin@cs.berkeley.edu>
U.C. Berkeley, EECS, CS Division

1 Introduction

Interactive 3D computer graphics extends beyond the creation and manipulation of fancy, virtual, on-screen models. It also plays an important role for making repetitive or symmetrical patterns, constructing 3D CAD files for rapid prototyping, or drawing 2D nets that can be folded up into 3D paper or cardboard models. To make effective mathematical visualization models one or more of these modes may have to be engaged. This will be demonstrated in the context of realizing highly symmetrical embeddings of regular maps on surfaces of higher genus.

2 Regular Maps

Regular maps are a generalization of periodic tilings. They are 2-manifold networks of vertices, edges, and faces, in which these elements are topologically indistinguishable from their siblings. The Platonic solids are examples of regular maps on surfaces of genus zero. For surfaces of higher genus we abandon the requirement of geometrical similarity of all tiles, and allow all tiles to stretch smoothly along the surface. The key property of a regular map is then its topological *flag transitivity*. This means that any element (vertex, edge, facet) can be moved onto any other such element, in any orientation, and the whole network can then be brought into coincidence with itself.

Regular maps are always subsets of an appropriate infinite hyperbolic tiling, which is best displayed on the Poincaré disk (Fig.1). But they are of finite extension, and only a subset of all the facets are needed to give complete coverage on a surface of finite genus. R5.13 is a regular map based on a {8,8} hyperbolic tiling, but it has only four octagons (shown in different colors in Figure 1), and eight facet corners join in every vertex.
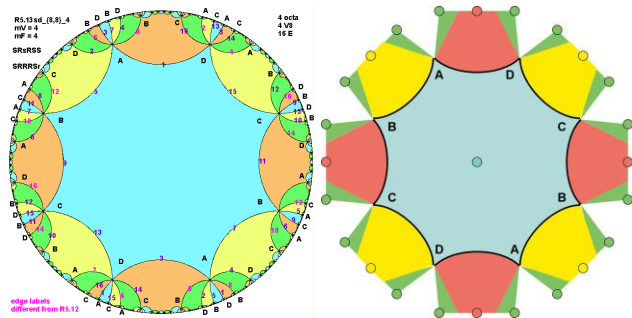

Figure 1. Poincare disk and fundamental net of R5.13_{8,8}.

There are finitely many regular maps on surfaces of genus 2 or higher. A brute-force computer search has enumerated all reflexive maps from genus 2 through genus 101 [Conder 2006]. There are 6104 of them, but until a couple of years ago only about a dozen of them had ever been depicted in a picture or a 3D model. While these maps can be seen as abstract, group-theoretical constructs, I wanted to see them as aesthetically pleasing, geometrical realizations of highest possible symmetry. This turned out to be much more challenging than first expected. In 2009 another brute-force computer search [VanWijk 2009] was able to find several dozen geometrical realizations of regular maps, some with a genus as high as 29. But due to some built-in assumptions and restrictions, that program failed to find solutions for many maps even on surfaces of low genus. Using interactive computer graphics in its many different roles, as well as tangible models made out of paper or Styrofoam, I was able to fill in all the missing models for genus 2 through 5 by the fall of 2010.

3 Finding Symmetrical Embeddings

The search for a highly symmetrical embedding of a particular regular map starts with a Poincaré disk and the identification of a fundamental net (Fig.1) and of the complete connectivity diagram of that map. Based on the dominant symmetries in that network, an educated guess then lets me select an overall symmetry group and a corresponding handle-body on which that network may be drawn. The set of vertices of the network is placed onto that surface in accordance with the chosen symmetry. They key challenge then is to place and connect up all edges of the network without any crossings. Eventually a computerized search-program may be developed that will automate this process, but in the meantime I have used the power of interactive 3D graphics to find solutions and to construct effective visualization models. Through several intermediated models (presented on the poster), I arrived at two final models (Fig.2), one a two-sided template formed into a toroidal ring, and the other a virtual tubular handle body.

Figure 2. Physical and virtual model of R5.13 [Séquin 2010].

4 Emerging Parameterized Patterns

A solution on a low-genus surface often implies solutions on surfaces of higher genus. A suitably parameterized model then allows to construct these models quickly by interactively adjusting a few parameters to yield pleasing and instructive models (Fig.3).
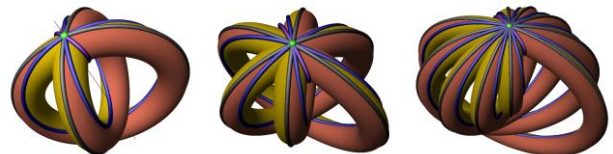

Figure 3. Maps R2.4_{5,10}, R3.9_{7,14}, R5.14_{11,22}.

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