Art and Math Behind and Beyond the 8-fold Way

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Abstract

The Eightfold Way is the name of a sculpture by Helaman Ferguson's at the Mathematical Sciences Research Institute in Berkeley. It celebrates Klein's *Quartic Curve*, a topologically regular configuration of 24 heptagons on a surface of genus 3. Various models of this intriguing mathematical objects are described, and attempts at finding other regular tilings on surfaces of higher genus are discussed.

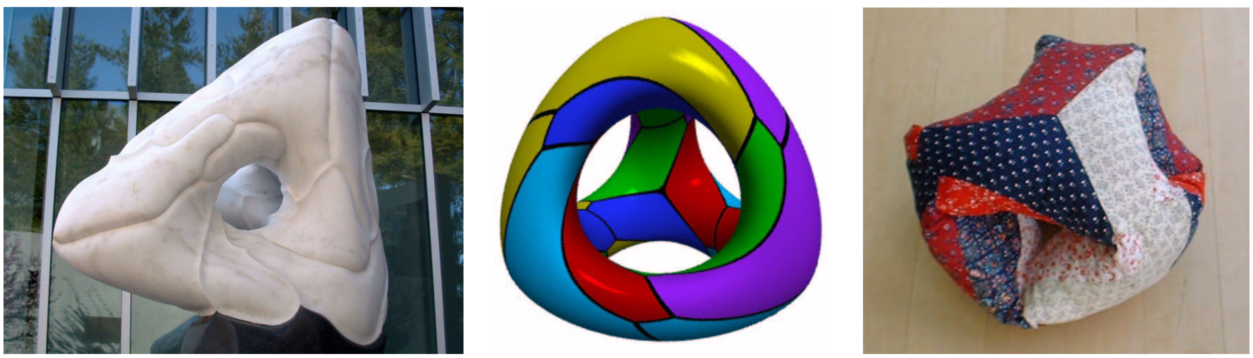

Figure 1: *The Eightfold Way: (a) sculpture by Helaman Ferguson at MSRI; (b) 24 heptagons on a "tetrus" – a smooth tetrahedral frame; (c) quilt made by Eveline Séquin.*

1. EIGHT-fold Ways

When thinking about an appropriate presentation for G4G8, something that would involve the number 8, some intriguing math or geometry to the point of appearing "magical", while also being aesthetically pleasing, I quickly came to think of the *Eightfold Way* [5]. There is a sculpture with that name at the Mathematical Sciences Research Institute (MSRI) in Berkeley (Fig.1a). It was created in 1993 by renowned sculptor and mathematician Helaman Ferguson to demonstrate and celebrate a very important and famous mathematical structure discovered in 1878 by Felix Klein [4], also known as Klein's *Quartic Curve*.

This marble sculpture forms a surface of genus 3, loosely resembling a rounded tetrahedral frame. On this "tetrus" surface, outlined by ridges and grooves, are depicted 24 heptagonal facets that join in groups of three at 56 vertices. This structure was given the name *The Eightfold Way* because of an intriguing property: An observer can put a finger at any vertex and move along consecutive edges between heptagons; if one takes alternatingly the left or the right branch at each subsequent junction of three edges, one will then end up where one started after exactly *eight* moves. These particular zig-zag paths are called *Petrie polygons*; they always hug any face for exactly two consecutive edges before moving away from it. They can also be drawn on the regular Platonic solids. On the tetrahedron the Petrie polygon has only four segments; on the cube and octahedron we obtain a "6-fold way", and on the dodecahedron and icosahedron we obtain zig-zag loops with 10 edges. On the Klein quartic all Petrie polygons are of length eight; the ones wrapping individually around each one of the tetrahedral arms are easy to follow, the other ones are trickier to trace without making a mistake.

But the connotations of the “8-fold Way” go deeper. When Helaman Ferguson selected the name for this scientific sculpture, he alluded to the use of the same term by some physicists in the 1960s, who worked on elementary particles. They used the term *Eightfold Way* to refer to some symmetries that they found among the rapidly growing family of newly discovered elementary particles [2]. In particular, a theory of strong-interaction symmetry lead to a classification of the known hadrons based on SU(3) symmetry, in which the neutron and proton were assigned to an 8-component family of related elementary particles. This innovative classification ultimately lead to the *Standard Model*, which unified most of the fundamental particles and forces

However, those Physicists in turn had borrowed the name from a much older concept. The *Noble Eightfold Path* [1] describes the way to end suffering as it was outlined by Siddhartha Gautama. It is a guide to ethical and mental development with the goal of freeing the individual from attachments and delusions. In the end it is supposed to lead to an understanding of the truth about all things.

But in this paper we start with the geometrical structure underlying Helaman Ferguson’s marble sculpture *The Eightfold Way;* then we continue to look for other regular tilings on surfaces of higher genus.

2. Locally and Globally Regular Tilings

The reason why the *Klein Quartic* is such an important mathematical structure is its high degree of topological symmetry [4]. The symmetry stems from the regularity of the mesh of heptagons. A tiling or a 2-manifold mesh is called *locally regular*, if all facets are *n*-gons with the same number of sides, and all vertices have the same valence, i.e., are shared by the same number of edges. Clearly the Klein Quartic is locally regular, since it consists of all heptagons, and three of them always join in a vertex. But topologically it also has the *global regularity* that we know from the Platonic solids: Any edge can be moved to any other edge (in either of its two orientations), and as we move the whole mesh correspondingly, every vertex, edge, and face will find a suitable vertex, edge, and face to land on. All vertices, edges, and faces are thus completely equivalent! In 4-dimensional space we could see this mesh as a completely regular structure in the same sense that the Platonic solids are completely regular polyhedral meshes. If we try to embed this construct in 3D, so that we can make a physical model of it, we lose most of its metric symmetries. The regular heptagons get distorted, and only the twelve symmetries of a regular tetrahedron with oriented faces are maintained (Fig.1b). However, the topological symmetries are fully preserved. Helaman Ferguson’s sculpture *The Eightfold Way* at MSRI in Berkeley celebrates this shape. A template for one of the 24 heptagons obtained from Bill Thurston allowed my daughter Eveline to stitch together the quilt shown in Figure 1c. This quilt also can be turned inside out, thus it doubles the number of symmetries present.

On a genus-3 surface there exist globally regular maps corresponding to the {7,3} tiling and to its dual the {3,7} tiling with 56 triangles. Both of these tilings have 168 *automorphisms* without reflections (ways of mapping back onto themselves). The Klein Quartic is so special because it exhibits the highest possible number of (topological) automorphisms of any map on a genus-3 surface. It turns out that for genus 2 and higher, no map on a surface of genus g can have more than $84(g\text{-}1)$ orientation-preserving automorphisms. This theoretical limit established by Hurwitz [3] cannot be achieved for all values of g. The first such value for which it can be achieved is g=3; the next one is g=7; and the next one after that is g=14. Thus I am particularly motivated to find a nice symmetrical embedding of the corresponding regular map with 84*(7-1)=504 automorphisms on a surface of genus 7.

3. Looking for a Globally Regular Tiling on a Surface of Genus 7

Perhaps we can take some inspiration from the genus-3 solution, the Klein Quartic, to make a model for the genus-7 case. If we study the various models in Figure 1, we can see that the heptagons are clustered in a very symmetrical manner around each of the four 3-way junctions forming the hubs of the tetrahedral frame underlying this surface. Six heptagons wrap around each junction, with two vertices placed at the two points of 3-fold symmetry (Fig.2a). The other vertices form zig-zag patterns around each of the six tetrahedral arms (along which we find some of the 8-segment Petrie polygons). A natural approach to forming a surface of genus 7 is to use the wire frame of a hexagonal prism or of an octahedron and then form the corresponding thickened tubular frames [6]. To build the hex-prism frame (Fig.2c), we need the same type of 3-way junction as for the tetrahedral frame (Fig.2a). But for the octahedral frame (Fig.2d) we need 4-way junctions. Topologically these latter junctions exhibit tetrahedral symmetry (Fig.2b), but to build an actual frame, they would then have to be bent substantially. However, this does not change the way that we plan to decorate these junctions with the heptagons. Around each such junction we might cluster 12 heptagons, again placing vertices at the points of 3-fold symmetry (of which there are now four). The serrated zig-zag polygons at the end of the arm stubs are now of length 12 (Fig.2b).

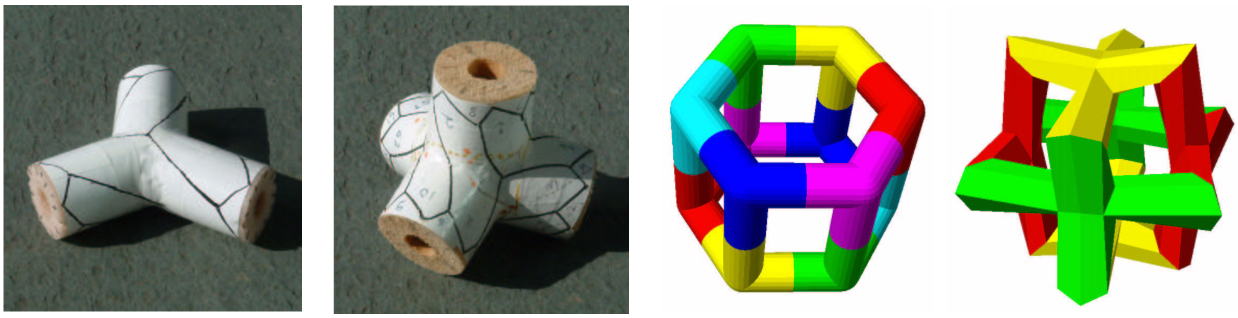

Figure 2: *Symmetrically decorated junctions: (a) 3-way, (b) 4-way junction. Two different ways of making genus-7 surfaces from such junction elements: (c) based on hex-prism, and (d) on octahedral frame.*

Fitting these junction elements together will definitely yield nice locally regular tilings of type {7,3}, but does it produce the sought-after global regularity? Checking an arbitrary mesh for global regularity is not a trivial task. But looking for the Petrie polygons, and making sure that all of them are of exactly the same length is a good test. On the *Klein Quartic* this corresponds to the *Eightfold Way* property.

Analyzing the topological properties of the Hurwitz mesh of maximal symmetry on a genus-7 surface, we find that we will need 72 heptagons, 168 vertices of valence 3, and Petrie polygons of length 18. While the first two requirements are readily fulfilled by the locally regular tiling resulting from joining together our modular components, the last one is not. One can clearly see that the length of the Petrie polygons formed around the arms of the octahedral frame are of length 12, instead of 18. Twisting the arms to try out different patterns of global connectivity does not change the length of these Petrie polygons either. Thus we need to look for a much more ingenious way of decorating these junction elements – or, perhaps, for an altogether different approach!

Looking for an embedding with the desired global regularity can be decomposed into two separate phases. First we just try to construct a graph (possibly with many crossings) that exhibits the proper connectivity of that sought-after object with 504 automorphisms; then we try to embed that graph in a surface of genus 7 with as much symmetry as possible. The first task can be carried out conveniently on the Poincaré disk, where we can readily display {7,3} and {3,7} tiling patterns with arbitrary many tiles. The crucial step is to find a periodically repetitive numbering of all tiles, so that a fundamental domain of exactly the right size is established, and this pattern then can be repeated indefinitely, while seamlessly covering the whole hyperbolic surface represented by the Poincaré disk. For the Klein quartic that domain comprises 24 heptagons [5], while for the genus-7 surface it comprises 72 heptagons (Fig.3a).

With the connectivity graph established, the harder part of the task can now be tackled. After having unsuccessfully worked with frames of octahedral symmetry, I started to focus on surfaces with a 7-fold symmetry. I have tried to map this graph onto disks with seven holes (Fig.3b) or onto cylinders with seven handles (Fig.3c). The paper models constructed for the latter case made it easy to experiment with various amounts of twist around those handles and around the main cylindrical body. Alas, no globally regular embedding has been found yet.

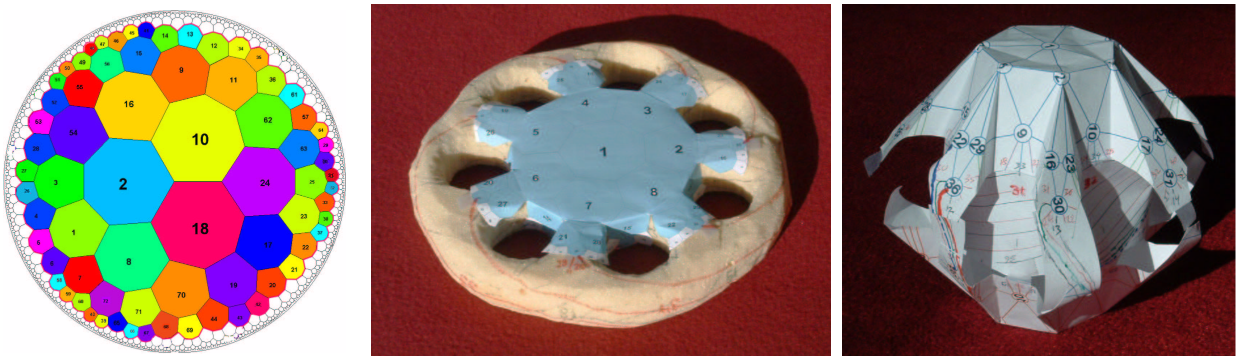

Figure 3: *Mapping the {7,3} tiling: (a) from* Poincaré *disk onto (b) a styrofoam disk with 7 tunnels, and (c) onto a paper model of a cylinder with 7 handles.*

4. Locally Regular Escher-like Tilings

Thus, for now, let's focus on a different task: to create intriguing and aesthetically pleasing tilings with only local regularity on surfaces of higher genus. In the past, my students have written programs to create Escher-like tilings on a torus or on a sphere [7]. In both cases, if one starts with the right symmetry constraints, all tiles can be identical. On the sphere one can have as many as 60 identical tiles. On the torus, one can have, in principle, infinitely many identical tiles, but they all must wrap around the toroidal arm in exactly the same way; this seriously limits the kind of tiles we can use for decoration. Similarly, on the higher-genus surfaces derived from either tetrahedral or octahedral frames one could have at most 12 or 24 identical tiles, and they all would have to cover some inside as well as outside portions of these tubular surfaces. If we want to use smaller, more localized tiles, we need to tolerate some modest distortions.

As a first example, each of the 24 heptagons on the *tetrus* (Fig.1b) has been replaced with an Escheresque lizard shape, thus leading to the *Lizard Tetrus* (Fig.4a), which graced the cover of the AMS *Calendar of Mathematical Imagery* in 2007. This mapping creates two different lizard shapes: a smaller one on the inside, and a larger one on the outside. Even disregarding the geometrical distortions due to the curvature of the surface, this tile does not honor the full regularity implied by the heptagonal tiles. To capture the full topological symmetry with 168 automorphisms, we divide each heptagon into seven identical wedges and distort the edges to create an interesting fish motif (Fig.4c). We maintain all symmetries of the fundamental domain, i.e., C_7 symmetry around the tile center, C_2 symmetry around edge midpoints, and C_3 symmetry around the vertices. These distorted tiles will then fit together to make a heptagonal tile (Plate 6a) that seamlessly covers the Poincaré disk (Plate 6b) and which can also be wrapped around the tetrus (Plate 6c).

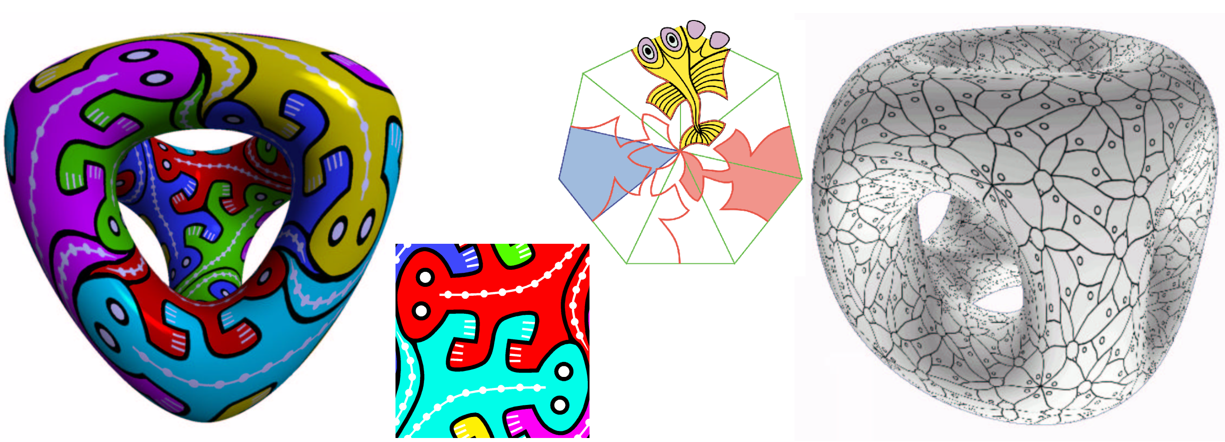

Figure 4: *Decorative regular tilings on higher-genus surfaces: (a) 24 lizards on the genus-3 tetrus; (b) texture tile with 2 lizards; (c) breaking up the hexagon; (d) 336 butterflies on a genus-5 surface.*

Since this display was created by texture mapping and I wanted to use only a single type of heptagonal tile, I had to select a very special coloring pattern and carefully assign the orientations for all heptagons, in order to have all colors properly match up at the tile borders. Note, that all yellow fish point towards the inner and outer tripodal poles of the tetrus. In this mapping most of the tiles get heavily distorted to achieve the sought-after globally regularity. Fortunately, Escher's "creature tiles" can tolerate such deformations.

Figure 4d shows an example of a tiling that exhibits local regularity only. Again we use a {7,3} tiling, but his time the heptagons are filled with 7 identical butterflies. This pattern is mapped onto a smooth cube frame of genus 5. There are a total of 336 butterflies on a genus-5 surface with the orientation-preserving symmetries of a cube (Fig.4d).

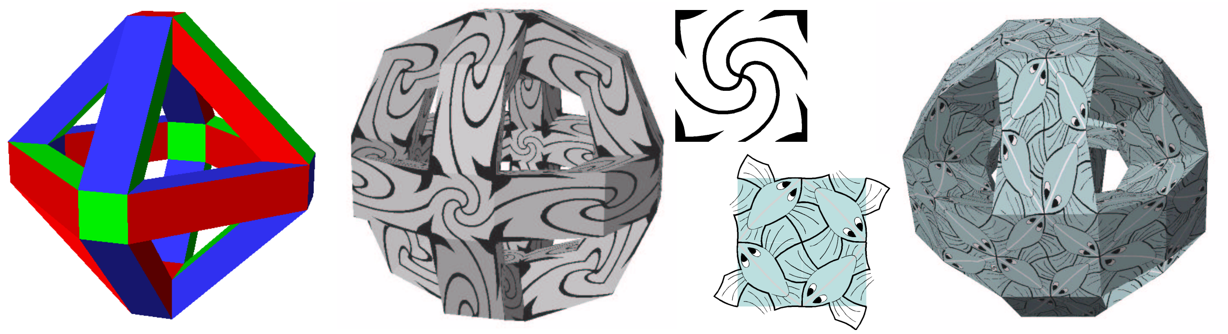

Figure 5: *Locally regular {4,5} tiling on a genus-7 surface: (a) octahedral frame composed of 60 quads; (b) surface cover with 60 swirls; (e) surface cover with 240 fish; (c, d) the two texture patterns used.*

Finally we construct some locally regular tilings on a genus-7 surface, derived from an octahedral frame. A polyhedral model of such a frame can be constructed from 60 quadrilaterals so that exactly 5 facets join together at every vertex (Fig.5a). Using a texture pattern with 4-fold symmetry (Fig.5c) makes it easy to cover the whole genus-7 surface with a seamless texture map (Fig.5b). If each quadrilateral is partitioned into four equivalent Escher fish (Fig.5d), a total of 240 such fish can then accommodated on this surface (Fig.5e).

Alternatively, if we re-color our spirally texture patterns with three different colors (Plate 7a) and then use the right color combinations and orientations on each quadrilateral, we can obtain a cohesive pattern of *5-arm starfish* that interlock with one another (Plate 7b). This now makes visible the dual {5,4} tiling pattern. We have converted the underlying polyhedral surface into a smooth Catmull-Clark subdivision surface (Plate 7c) and then extracted the geometry of two proto-tiles: an inner (yellow) and an outer (blue) starfish tile (Plate 7d). The geometries of these two tiles were thickened by using offset surfaces, and the resulting solid models were sent to a Fused Deposition Modeling (FDM) rapid prototyping machine. Four tiles of each type were made in six different colors, and these physical tiles were then glued together (Plate 8a-c), creating a genus-7 object composed of 48 starfish tiles.

References

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COLOR PLATES

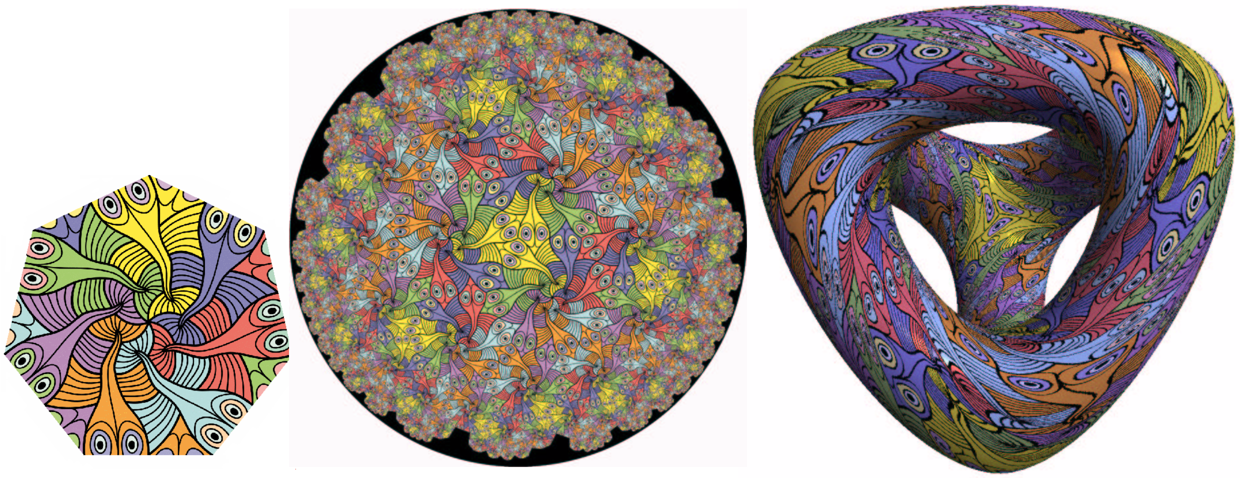

Plate 6: *{7,3} tiling with fish pattern of Fig.4a: (a) one heptagonal tile; (b) infinite tiling on Poincaré disk; (c) 168 fish on a genus-3 tetrus surface.*

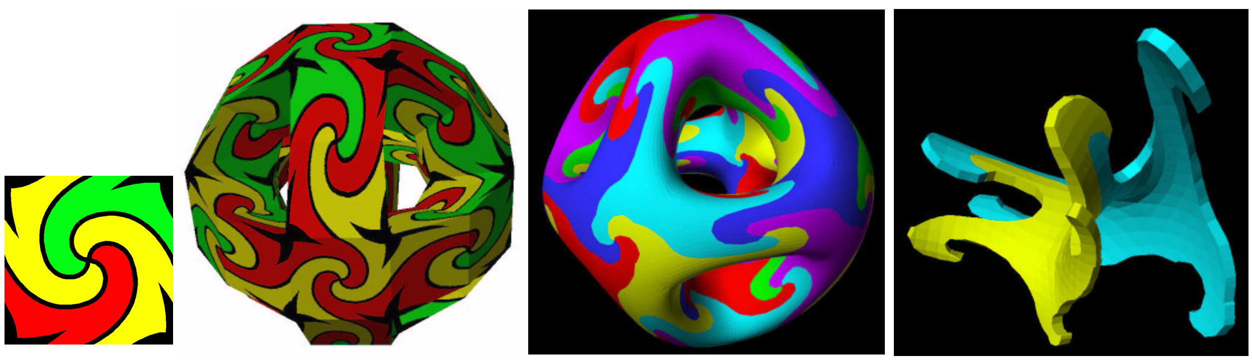

Plate 7: *{5,4} starfish tiling: (a) one colored quadrilateral tile; (b) texture-mapped onto polyhedron of genus 7 with 60 quads; (c) converted into a smooth subdivision surface; (d) two proto-tiles extracted.*

Plate 8: *Assembly of starfish tiles: (a) one arm of octahedral frame; (b) assembly about half done; (c) octahedral genus-7 surface with 48 starfish tiles.*