Differential Geometry of Surfaces

Jordan Smith and Carlo Séquin*
CS Division, UC Berkeley

1 Introduction

These are notes on differential geometry of surfaces based on reading [Greiner et al. n. d.].

2 Differential Geometry of Surfaces

Differential geometry of a 2D manifold or surface embedded in 3D is the study of the intrinsic properties of the surface as well as the effects of a given parameterization on the surface. For the discussion below, we will consider the local behavior of a surface S at a single point p with a given local parameterization (u, v), see Figure 1. We would like to calculate the principle curvature values (κ_1, κ_2), the maximum and minimum curvature values at p, as well as the local orthogonal arc length parameterization (s, t) which is aligned with the directions of maximum and minimum curvature.

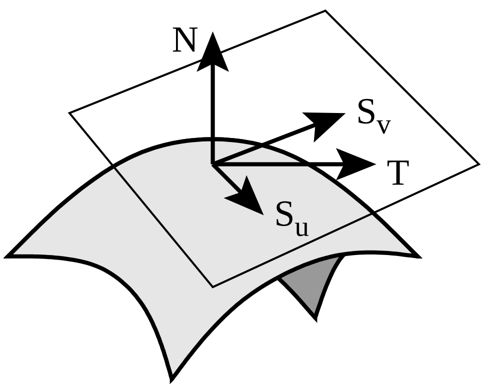


Figure 1: *Tangent plane of S*

3 The First Fundamental Form

With the given parameterization, we can compute a pair of tangent vectors $\left[\begin{array}{cc} S_u & S_v \end{array}\right] = \left[\begin{array}{cc} \frac{\partial S}{\partial u} & \frac{\partial S}{\partial v} \end{array}\right]$ assuming the u and v directions are distinct. These two vectors locally parameterize the tangent plane to S at p. A general tangent vector T can be constructed as a linear combination of $\left[\begin{array}{cc} S_u & S_v \end{array}\right]$ scaled by infinitesimal coefficients $U = \left[\begin{array}{cc} \partial u & \partial v \end{array}\right]^t$.

$$T = \partial u S_u + \partial v S_v = \left[\begin{array}{cc} S_u & S_v \end{array}\right] \left[\begin{array}{c} \partial u \\ \partial v \end{array}\right] \quad (1)$$

Consider rotating the direction of the tangent T around p by setting $\left[\begin{array}{cc} \partial u & \partial v \end{array}\right] = \left[\begin{array}{cc} \cos\theta & \sin\theta \end{array}\right]$. In general, the magnitude and direction of T will vary based on the skew and scale of the basis vectors $\left[\begin{array}{cc} S_u & S_v \end{array}\right]$, see Figure 3(a). The distortion of the local parameterization is described by the metric tensor or the first fundamental form I_S. This measures the length of a tangent vector on a given parametric basis by examining the quantity $T \cdot T$.

$$\begin{aligned}
T \cdot T &= (S_u \cdot S_u)\,\partial u^2 + 2\,(S_u \cdot S_v)\,\partial u \partial v + (S_v \cdot S_v)\,\partial v^2 && (2)\\
&= E\partial u^2 + 2F\partial u\partial v + G\partial v^2 && (3)\\
&= \left[\begin{array}{cc} \partial u & \partial v \end{array}\right] \left[\begin{array}{c} S_u \\ S_v \end{array}\right] \cdot \left[\begin{array}{cc} S_u & S_v \end{array}\right] \left[\begin{array}{c} \partial u \\ \partial v \end{array}\right] && (4)\\
&= \left[\begin{array}{cc} \partial u & \partial v \end{array}\right] \left[\begin{array}{cc} S_u \cdot S_u & S_u \cdot S_v \\ S_u \cdot S_v & S_v \cdot S_v \end{array}\right] \left[\begin{array}{c} \partial u \\ \partial v \end{array}\right] && (5)\\
&= \left[\begin{array}{cc} \partial u & \partial v \end{array}\right] \left[\begin{array}{cc} E & F \\ F & G \end{array}\right] \left[\begin{array}{c} \partial u \\ \partial v \end{array}\right] && (6)\\
&= U^t I_S U && (7)
\end{aligned}$$

Equation 7 defines I_S.

$$\begin{aligned}
I_S = g_{ij} &= \left[\begin{array}{c} S_u \\ S_v \end{array}\right] \cdot \left[\begin{array}{cc} S_u & S_v \end{array}\right] && (8)\\
&= \left[\begin{array}{cc} S_u \cdot S_u & S_u \cdot S_v \\ S_u \cdot S_v & S_v \cdot S_v \end{array}\right] && (9)\\
&= \left[\begin{array}{cc} E & F \\ F & G \end{array}\right] && (10)
\end{aligned}$$

If an orthonormal arc length parameterization (s, t) is chosen then there will be no local metric distortion and I_S will reduce to the identity matrix.

4 The Second Fundamental Form

The unit normal N of a surface S at p is the vector perpendicular to S, i.e. the tangent plane of S, at p. N can be calculated given a general nondegenerate parametrization (u, v).

$$N = \frac{S_u \times S_v}{\|S_u \times S_v\|} \quad (11)$$

The curvature of a surface S at a point p is defined by the rate that N rotates in response to a unit tangential displacement as in Figure 2. A normal section curve at p is constructed by intersecting S with a plane normal to it, i.e a plane that contains N and a tangent direction T. The curvature of this curve is the curvature of S in the direction T. The curvature κ of a curve is the reciprocal of the radius ρ of the best fitting osculating circle, i.e. $\kappa = \frac{1}{\rho}$. The directional derivative N_T of N in the direction T is a linear combination of the partial derivative vectors $\left[\begin{array}{cc} N_u & N_v \end{array}\right] = \left[\begin{array}{cc} \frac{\partial N}{\partial u} & \frac{\partial N}{\partial v} \end{array}\right]$ and is parallel to the tangent plane.

*e-mail: (jordans|sequin)@cs.berkeley.edu

$$N_u = \frac{(S_{uu} \times S_v) + (S_u \times S_{vu})}{\|S_u \times S_v\|} + (S_u \times S_v)\,\frac{\partial \|S_u \times S_v\|^{-1}}{\partial u} \tag{12}$$

$$N_v = \frac{(S_{uv} \times S_v) + (S_u \times S_{vv})}{\|S_u \times S_v\|} + (S_u \times S_v)\,\frac{\partial \|S_u \times S_v\|^{-1}}{\partial u} \tag{13}$$

$$N_T = \partial u N_u + \partial v N_v = \begin{bmatrix} N_u & N_v \end{bmatrix} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} \tag{14}$$

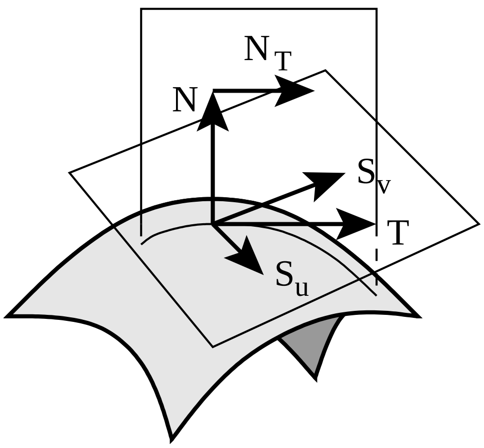


Figure 2: *Normal section of S*

As with the tangent vector T, N_T is subject to scaling by the metric of the parameter space. In an arc length parameter direction s, $-S_s \cdot N_s$ is the curvature of the normal section. Given a general parameterization (u, v), we can compute $-T \cdot N_T$. This quantity defines the second fundamental form II_S which describes curvature information distorted by the local metric.

$$\begin{aligned} -T \cdot N_T &= -(S_u \cdot N_u)\,\partial u^2 - (S_u \cdot N_v + S_v \cdot N_u)\,\partial u \partial v \\ &\quad - (S_v \cdot N_v)\,\partial v^2 && (15) \\ &= \begin{bmatrix} \partial u & \partial v \end{bmatrix} \begin{bmatrix} -S_u \\ -S_v \end{bmatrix} \cdot \begin{bmatrix} N_u & N_v \end{bmatrix} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} && (16) \\ &= \begin{bmatrix} \partial u & \partial v \end{bmatrix} \begin{bmatrix} -S_u \cdot N_u & -S_u \cdot N_v \\ -S_v \cdot N_u & -S_v \cdot N_v \end{bmatrix} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} && (17) \\ &= U^t II_S U && (18) \end{aligned}$$

Equation 18 defines II_S. II_S can be simplified to a set of quanties which are easier to compute by substituting the expressions in Equations [12,13] for N_u and N_v respectively. Two of the terms of in of the entries of the matrix in Equation 21 vanish due to the dot product of orthogonal vectors. Further simplification using properties of the box product of vectors yields Equation 22. Equation 23 is derived by subsituting N from Equation 11 and shows that II_S can be computed simply by the dot product of the second partial derivatives of S with N. Equation 23 also demonstrates that II_S is a symmetric matrix.

$$\begin{aligned} II_S = h_{ij} &= \begin{bmatrix} S_u \\ S_v \end{bmatrix} \cdot \begin{bmatrix} -N_u & -N_v \end{bmatrix} && (19) \\ &= \begin{bmatrix} -S_u \cdot N_u & -S_u \cdot N_v \\ -S_v \cdot N_u & -S_v \cdot N_v \end{bmatrix} && (20) \\ &= \begin{bmatrix} -\frac{S_u \cdot (S_{uu} \times S_v)}{\|S_u \times S_v\|} & -\frac{S_u \cdot (S_{uv} \times S_v)}{\|S_u \times S_v\|} \\ -\frac{S_v \cdot (S_u \times S_{vu})}{\|S_u \times S_v\|} & -\frac{S_v \cdot (S_u \times S_{vv})}{\|S_u \times S_v\|} \end{bmatrix} && (21) \\ &= \begin{bmatrix} \frac{S_{uu} \cdot (S_u \times S_v)}{\|S_u \times S_v\|} & \frac{S_{uv} \cdot (S_u \times S_v)}{\|S_u \times S_v\|} \\ \frac{S_{vu} \cdot (S_u \times S_v)}{\|S_u \times S_v\|} & \frac{S_{vv} \cdot (S_u \times S_v)}{\|S_u \times S_v\|} \end{bmatrix} && (22) \\ &= \begin{bmatrix} S_{uu} \cdot N & S_{uv} \cdot N \\ S_{uv} \cdot N & S_{vv} \cdot N \end{bmatrix} && (23) \\ &= \begin{bmatrix} L & M \\ M & N \end{bmatrix} && (24) \end{aligned}$$

Equations [23,24] can be substituted for II_S to yield alternate formulas for $-T \cdot N_T$.

$$\begin{aligned} -T \cdot N_T &= (S_{uu} \cdot N)\,\partial u^2 + 2\,(S_{uv} \cdot N)\,\partial u \partial v + (S_{vv} \cdot N)\,\partial v^2 && (25) \\ &= L \partial u^2 + 2M \partial u \partial v + N \partial v^2 && (26) \\ &= \begin{bmatrix} \partial u & \partial v \end{bmatrix} \begin{bmatrix} S_{uu} \cdot N & S_{uv} \cdot N \\ S_{vu} \cdot N & S_{vv} \cdot N \end{bmatrix} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} && (27) \\ &= \begin{bmatrix} \partial u & \partial v \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} && (28) \end{aligned}$$

II_S contains information about the curvature of S, but it will be distorted by the metric if the given parameterization is not an arc length parameterization. It is still necessary to counter act this distortion in order to compute the curvature of the S.

5 Coordinate Transformations

Our purpose in studying differential geometry and the fundamental forms has been to compute the principle curvature values (κ_1, κ_2) and directions (S_{k1}, S_{k2}) of a parametric surface S at a point p. Thus far we have derived II_S which contains curvature information but can be distorted by the metric of the parameterization. I_S describes this metric information. We must combine II_S and I_S to find the arc length curvature values.

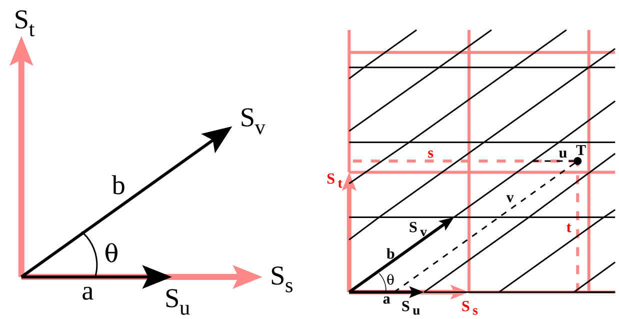
The given parameterization (u, v) defines a set of basis tangent vectors $\begin{bmatrix} S_u & S_v \end{bmatrix}$. We can construct an orthonormal basis $\begin{bmatrix} S_s & S_t \end{bmatrix}$ due to an arc length parameterization (s, t) where S_s and S_u are aligned, see Figure 3(a). A point T can be expressed in either coordinate system, see Figure 3(b).

$$T = \begin{bmatrix} \partial u & \partial v \end{bmatrix} \begin{bmatrix} S_u \\ S_v \end{bmatrix} = \begin{bmatrix} \partial s & \partial t \end{bmatrix} \begin{bmatrix} S_s \\ S_t \end{bmatrix} \tag{29}$$

The coordinates $\begin{bmatrix} \partial s & \partial t \end{bmatrix}$ measure the length of T as opposed to the coordinates $\begin{bmatrix} \partial u & \partial v \end{bmatrix}$.

$$T \cdot T = \partial s^2 + \partial t^2 \neq \partial u^2 + \partial v^2 \tag{30}$$

We would like to work in the $\begin{bmatrix} S_s & S_t \end{bmatrix}$, so we must find the transformations to and from $\begin{bmatrix} S_u & S_v \end{bmatrix}$. The quantities a, b, and θ in Figure 3(a) are defined by $\begin{bmatrix} S_u & S_v \end{bmatrix}$.



(a) *Constructing an orthonormal basis on top of the given parametric basis*

(b) *A point T in (u, v) and (s, t) coordinates*

Figure 3: *Effects of the metric*

$$\begin{bmatrix} S_u \\ S_v \end{bmatrix} = \begin{bmatrix} a & 0 \\ b\cos\theta & b\sin\theta \end{bmatrix} \begin{bmatrix} S_s \\ S_t \end{bmatrix} \tag{31}$$

$$= A \begin{bmatrix} S_s \\ S_t \end{bmatrix} \tag{32}$$

$$\begin{bmatrix} S_s \\ S_t \end{bmatrix} = \frac{1}{ab\sin\theta} \begin{bmatrix} b\sin\theta & 0 \\ -b\cos\theta & a \end{bmatrix} \begin{bmatrix} S_u \\ S_v \end{bmatrix} \tag{33}$$

$$= A^{-1} \begin{bmatrix} S_u \\ S_v \end{bmatrix} \tag{34}$$

The first fundamental form I_S can be represented by these coordinate transformations.

$$I_S = \begin{bmatrix} S_u \\ S_v \end{bmatrix} \cdot \begin{bmatrix} S_u & S_v \end{bmatrix} \tag{35}$$

$$= A \begin{bmatrix} S_s \\ S_t \end{bmatrix} \cdot \begin{bmatrix} S_s & S_t \end{bmatrix} A^t \tag{36}$$

$$= A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^t \tag{37}$$

$$= AA^t \tag{38}$$

$$= \begin{bmatrix} a^2 & ab\cos\theta \\ ab\cos\theta & b^2 \end{bmatrix} \tag{39}$$

We can also derive transformations between coordinates of the different sets of basis vectors by substituting Equations [32,34] respectively into Equation 29.

$$\begin{bmatrix} \partial s & \partial t \end{bmatrix} = \begin{bmatrix} \partial u & \partial v \end{bmatrix} A \tag{40}$$

$$\begin{bmatrix} \partial u & \partial v \end{bmatrix} = \begin{bmatrix} \partial s & \partial t \end{bmatrix} A^{-1} \tag{41}$$

To verify that this makes sense, consider $T \cdot T$ again and transform the coordinates using Equations [40,41].

$$T \cdot T = \begin{bmatrix} \partial u & \partial v \end{bmatrix} \begin{bmatrix} S_u \\ S_v \end{bmatrix} \cdot \begin{bmatrix} S_u & S_v \end{bmatrix} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} \tag{42}$$

$$= \begin{bmatrix} \partial u & \partial v \end{bmatrix} I_S \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} \tag{43}$$

$$= \begin{bmatrix} \partial s & \partial t \end{bmatrix} A^{-1} I_S \left(A^{-1}\right)^t \begin{bmatrix} \partial s \\ \partial t \end{bmatrix} \tag{44}$$

$$= \begin{bmatrix} \partial s & \partial t \end{bmatrix} A^{-1} \left(AA^t\right) \left(A^{-1}\right)^t \begin{bmatrix} \partial s \\ \partial t \end{bmatrix} \tag{45}$$

$$= \begin{bmatrix} \partial s & \partial t \end{bmatrix} \begin{bmatrix} \partial s \\ \partial t \end{bmatrix} \tag{46}$$

$$= \partial s^2 + \partial t^2 \tag{47}$$

It is now possible to transform our given parameters to coordinates on an orthonormal basis where we can measure lengths. We can also transform coordinates on the orthonormal basis back to coordinates on our given parameterization.

6 Curvature

The curvature at a point p on the surface S in any tangential direction T is $-T \cdot N_T$.

$$-T \cdot N_T = \begin{bmatrix} \partial u & \partial v \end{bmatrix} \begin{bmatrix} S_u \\ S_v \end{bmatrix} \cdot \begin{bmatrix} -N_u & -N_v \end{bmatrix} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} \tag{48}$$

$$= \begin{bmatrix} \partial u & \partial v \end{bmatrix} II_S \begin{bmatrix} \partial u \\ \partial v \end{bmatrix} \tag{49}$$

This function is parameterized by the given parameterization. We would like to study the function that sweeps a unit length tangent around p, so it is more convenient to use arc length coordinates. We must transform coordinates so that setting the arc length coordinates $\begin{bmatrix} \partial s & \partial t \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \end{bmatrix}$ sweeps out the curvature values.

$$-T \cdot N_T = \begin{bmatrix} \partial s & \partial t \end{bmatrix} A^{-1} II_S \left(A^{-1}\right)^t \begin{bmatrix} \partial s \\ \partial t \end{bmatrix} \tag{50}$$

$$= \begin{bmatrix} \partial s & \partial t \end{bmatrix} II_{\hat{S}} \begin{bmatrix} \partial s \\ \partial t \end{bmatrix} \tag{51}$$

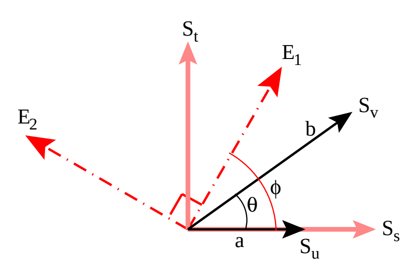


Figure 4: *The relationship between the principle parameter directions (E_1, E_2), the given general parameter directions (S_u, S_v), and the constructed orthonormal basis (S_s, S_t)in the tangent plane to S at p*

The curvature function will have maximum and minimum values that occur at directions that are orthogonal to each other. The eigenvalues of $II_{\hat{S}}$ are the principle curvature values (κ_1, κ_2). The eigenvectors of $II_{\hat{S}}$ are the coordinates of the principle curvature directions in the arc length coordinates. These coordinates will not be directly useful because we only know the $\left[\begin{array}{cc} S_u & S_v \end{array} \right]$ vectors, so we would actually prefer the eigenvectors in coordinates on this basis.

$$II_{\hat{S}} = A^{-1} II_S \left(A^{-1}\right)^t \tag{52}$$

$$= \frac{1}{a^2b^2\sin^2\theta} \left[\begin{array}{cc} b\sin\theta & 0 \\ -b\cos\theta & a \end{array} \right] \left[\begin{array}{cc} L & M \\ M & N \end{array} \right] \left[\begin{array}{cc} b\sin\theta & -b\cos\theta \\ 0 & a \end{array} \right] \tag{53}$$

$$= \frac{1}{a^2b^2\sin^2\theta} \cdot \left[\begin{array}{cc} b^2\sin^2\theta & b\sin\theta\left(-b\cos\theta L + aM\right) \\ b\sin\theta\left(-b\cos\theta L + aM\right) & b^2\cos^2\theta L - 2ab\cos\theta M + a^2 N \end{array} \right] \tag{54}$$

Solving for the eigenvalues, we find the characteristic equation. The principle curvature values are the eigenvalues.

$$0 = \left(a^2b^2 - (ab\cos\theta)^2\right)\lambda^2 - \left(b^2L - 2ab\cos\theta M + a^2N\right)\lambda + \left(LN - M^2\right) \tag{55}$$

$$0 = \left(EG - F^2\right)\lambda^2 - \left(GL - 2FM + EN\right)\lambda + \left(LN - M^2\right) \tag{56}$$

7 The Weingarten Operator

An alternate way of computing the principle curvature values and directions is by using the Weingarten Operator W, also known as the shape operator. W is the inverse of the first fundamental form multiplied by the second fundamental form.

$$W = I_S^{-1} II_S \tag{57}$$

I_S^{-1} is the inverse of the first fundamental form.

$$I_S^{-1} = \left(AA^t\right)^{-1} = \left(A^{-1}\right)^t A^{-1} \tag{58}$$

$$= \frac{1}{EG - F^2} \left[\begin{array}{cc} G & -F \\ -F & E \end{array} \right] \tag{59}$$

We will prove that W has the same eigenvalues as $II_{\hat{S}}$, i.e. the same principle curvature values. We will also prove that the eigenvectors of W are transformed versions of the eigenvectors of $II_{\hat{S}}$ to coordinates on the given basis $\left[\begin{array}{cc} S_u & S_u \end{array} \right]$.

$$II_{\hat{S}} = V\Lambda V^{-1} \tag{60}$$

$$II_S = A II_{\hat{S}} A^t \tag{61}$$

$$W = I_S^{-1} II_S \tag{62}$$

$$= \left(A^{-1}\right)^t A^{-1} II_S \tag{63}$$

$$= \left(A^{-1}\right)^t A^{-1} A II_{\hat{S}} A^t \tag{64}$$

$$= \left(A^{-1}\right)^t II_{\hat{S}} A^t \tag{65}$$

$$W = \left(A^{-1}\right)^t V\Lambda V^{-1} A^t \tag{66}$$

Equation 66 is the eigendecomposition of W. The diagonal matrix Λ has the eigenvalues of W on its diagonal. It is the same eigenvalue matrix as for $II_{\hat{S}}$, so the eigenvalues of W are the principle curvature values. $\left(A^{-1}\right)^t V = \left(V^{-1}A^t\right)^- 1$, so the columns of $\left(A^{-1}\right)^t V$ are the eigenvectors of W. This is a transformed version of V, the eigenvectors of $II_{\hat{S}}$ over the arc length basis. The transformation A^{-1} transforms the arc length coordinates to coordinates over the given basis, see Equation 41.

$$V = \left[\begin{array}{cc} s_1 & s_2 \\ t_1 & t_2 \end{array} \right] \tag{67}$$

$$\left(A^{-1}\right)^t V = \left(A^{-1}\right)^t \left[\begin{array}{cc} s_1 & s_2 \\ t_1 & t_2 \end{array} \right] \tag{68}$$

$$= \left[\begin{array}{cc} u_1 & u_2 \\ v_1 & v_2 \end{array} \right] \tag{69}$$

To solve for the principle curvature values and directions using the given parameterization, we substitute Equations [24,59] into Equation 57 and simplify to derive W written as the coefficients of the first and second fundamental forms.

$$W = I_S^{-1} II_S \tag{70}$$

$$= \frac{1}{EG - F^2} \left[\begin{array}{cc} G & -F \\ -F & E \end{array} \right] \left[\begin{array}{cc} L & M \\ M & N \end{array} \right] \tag{71}$$

$$= \frac{1}{EG - F^2} \left[\begin{array}{cc} GL - FM & GM - FN \\ EM - FL & EN - FM \end{array} \right] \tag{72}$$

Then we use the representation of W in Equation 72 and compute the eigendecomposition. The eigenvalues are the principle curvatures, and the eigenvectors are the coordinate coefficients over the given parametric basis vectors.

$$\kappa_{1,2} = \lambda_{1,2} = \frac{GL - 2FM + EN}{2\left(EG - F^2\right)} \mp \frac{\sqrt{\left(GL - 2FM + EN\right)^2 - 4\left(EG - F^2\right)\left(LN - M^2\right)}}{2\left(EG - F^2\right)} \tag{73}$$

$$\kappa_M = \frac{\kappa_1 + \kappa_2}{2} = \frac{GL - 2FM + EN}{2\left(EG - F^2\right)} = \frac{Trace\left(W\right)}{2} \tag{74}$$

$$\kappa_G = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2} = Det\left(W\right) \tag{75}$$

References

GREINER, G., LOOS, J., AND WESSELINK, W. Data Dependent Thin Plate Energy and its use in Interactive Surface Modeling.

A Square Matrices: A Quick Reference

A.1 General 2×2 Matrices

We will be manipulating the 2×2 matrices I_S and II_S, so it is important to review some fundamental properties. Matrix B in Equation A-1 is a general 2×2 matrix. The inverse, B^{-1}, can be computed in closed form as shown in Equation A-2, if the determinant of B, $Det\left(B\right) = \frac{(ad-bc)}{\alpha}$, does not equal to zero.

$$B = \frac{1}{\alpha} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{(A-1)}$$

$$B^{-1} = \frac{\alpha}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(A-2)}$$

A.2 Eigenvalues

For a general $n \times n$ square matrix A, the eigenvalues of A are n special scalar values $\lambda_i \left(i : 1, n\right)$ such that for special corresponding eigenvectors $v_i \left(i : 1, n\right)$, multiplication of A with v_i is the same as scaling v_i by λ_i, Equation A-3. By substituting v_i with itself multiplied by the identity matrix I in Equation A-4, we can then combine terms in Equation A-5.

$$\lambda_i v_i = A v_i \quad \text{(A-3)}$$
$$\lambda_i \left(I v_i\right) = A v_i \quad \text{(A-4)}$$
$$\lambda_i I v_i = A v_i$$
$$0 = \left(A - \lambda_i I\right) v_i \quad \text{(A-5)}$$

v_i is in the nullspace of the matrix $\left(A - \lambda_i I\right)$, but the vector $v_i = 0$ always satisfies Equation A-5. We are interested in nonzero eigenvectors v_i only, so $\left(A - \lambda_i I\right)$ must be singular so that its nullspace will be nonempty. $\left(A - \lambda_i I\right)$ is singular if its determinant is zero. Equation A-6 is the characteristic equation of A, and the n roots of this equation define the eigenvalues $\lambda_i \left(i : 1, n\right)$.

$$0 = Det\left(A - \lambda_i I\right) \quad \text{(A-6)}$$

For the case of the general 2×2 matrix B, the characteristic equation, Equation A-7 can be expanded analytically to yield an equation that is quadratic in λ_i, Equation A-11.

$$0 = Det\left(B - \lambda_i I\right) \quad \text{(A-7)}$$

$$0 = Det\left(\frac{1}{\alpha} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{bmatrix}\right) \quad \text{(A-8)}$$

$$0 = \frac{1}{\alpha} Det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \alpha\lambda_i & 0 \\ 0 & \alpha\lambda_i \end{bmatrix}\right) \quad \text{(A-9)}$$

$$0 = \frac{1}{\alpha} \begin{vmatrix} a - \alpha\lambda_i & b \\ c & d - \alpha\lambda_i \end{vmatrix} \quad \text{(A-10)}$$

$$0 = \frac{1}{\alpha} \left(\alpha^2 \lambda_i^2 - \alpha\left(a + d\right)\lambda_i - \left(ad - bc\right)\right) \quad \text{(A-11)}$$

Equation A-11 can be solved analytically using the quadratic equation to yield the 2 eigenvalues λ_1 and λ_2 in Equation A-12.

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{\left(a - d\right)^2 + 4bc}}{2\alpha} \quad \text{(A-12)}$$

A.3 Eigenvectors

For a general $n \times n$ square matrix A, once the eigenvalues $\lambda_i \left(i : 1, n\right)$ have been computed, the corresponding eigenvectors $v_i \left(i : 1, n\right)$ are computed by finding the nullspace of the matrix $\left(A - \lambda_i I\right)$. The null space is computed for each λ_i by substituting its value into the matrix $\left(A - \lambda_i I\right)$ and then performing Gaussian elimination. The matrix will be singular, so at least one row will become all zeroes. It is then possible to solve for the components of v_i as parametric equations of a subset of free components.

The parametric nature of the v_i's show that the eigenvectors are not unique. In Equation A-3, v_i can be replaced with itself multiplied by an abitrary scalar. This arbitrary scaling of the eigenvectors is the only degree of freedom if the eigenvalues are unique, i.e. their algebraic multiplicities are all one. In this case, the nullspaces are all 1-dimensional subspaces, lines through the origin.

If there are multiple roots to the characteristic equation, then the algebraic multiplicity m of some of the eigenvalues will be greater than one. The m eigenvectors corresponding to such an eigenvalue will be a set of linearly independent vectors that span the m-dimensional subspace. The choice of these vectors has more degrees of freedom than the scaling in the 1-dimension case. If $m = 2$, then the subspace is a plane through the origin, and there is a rotational degree of freedom for choosing the eigenvectors as well as the scaling degree of freedom.

In general, there are some matrices with eigenvalues of algebraic multiplicity $m > 1$ that do not have a complete set of m associated eigenvectors, i.e. the geometric multiplicity is less than the algebraic multiplicity. These matrices are defective. For our purposes for the case of 2×2 matrices, we will assume that the matrices are non-defective.

For the case of a non-defective 2×2 matrix B, we will try to solve for the eigenvectors analytically by substituting the eigenvalues from Equation A-12 into the matrix in Equation A-10 and then performing Gaussian elimination.

$$0 = \frac{1}{\alpha} \begin{bmatrix} a - \alpha\lambda_i & b \\ c & d - \alpha\lambda_i \end{bmatrix} v_i \quad \text{(A-13)}$$

$$0 = \begin{bmatrix} a - \alpha\lambda_i & b \\ c & d - \alpha\lambda_i \end{bmatrix} \begin{bmatrix} v_{i,x} \\ v_{i,y} \end{bmatrix} \quad \text{(A-14)}$$

$$0 = \begin{bmatrix} a - \frac{a+d\pm\sqrt{(a-d)^2+4bc}}{2} & b \\ c & d - \frac{a+d\pm\sqrt{(a-d)^2+4bc}}{2} \end{bmatrix} \begin{bmatrix} v_{i,x} \\ v_{i,y} \end{bmatrix} \quad \text{(A-15)}$$

$$0 = \begin{bmatrix} \frac{(a-d)\mp\sqrt{(a-d)^2+4bc}}{2} & b \\ c & \frac{-(a-d)\mp\sqrt{(a-d)^2+4bc}}{2} \end{bmatrix} \begin{bmatrix} v_{i,x} \\ v_{i,y} \end{bmatrix} \quad \text{(A-16)}$$

$$0 = \begin{bmatrix} 0 & 0 \\ 1 & \frac{-(a-d)\mp\sqrt{(a-d)^2+4bc}}{2c} \end{bmatrix} \begin{bmatrix} v_{i,x} \\ v_{i,y} \end{bmatrix} \quad \text{(A-17)}$$

Equation A-17 shows that the matrix $\left(A - \lambda_i I\right)$ is singular. We will let $v_{i,y} = 1$ then we can solve for $v_{i,y}$, Equation A-18.

$$v_{1,2} = \begin{bmatrix} \frac{(a-d)\pm\sqrt{(a-d)^2+4bc}}{2c} \\ 1 \end{bmatrix} \quad \text{(A-18)}$$

In Equation A-18, if $c = 0$, then the eigenvector is not well defined. We must handle the case where $c = 0$ or $b = 0$ separately. In this case the eigenvalues simplify to $\lambda_1 = \frac{a}{\alpha}$ and $\lambda_2 = \frac{d}{\alpha}$, and gaussian elimination will produce a matrix with a single nonzero element. Hence, $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^t$ and $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^t$, respectively.

A.4 Eigen Decomposition

For a non-defective $n \times n$ square matrix A, the geometric multiplicity equals the algebraic multiplicity for all of the eigenvalues $\lambda_i\,(i:1,n)$, so there is a corresponding set of n linearly independent eigenvectors $v_i\,(i:1,n)$. It is then possible to diagonalize A by transforming A by the eigenvectors and their inverse. Remembering Equation A-19 that defines each eigenvector, we can write all n equations as columns next to each other in Equation A-20.

$$Av_i = v_i\lambda_i \tag{A-19}$$

$$A\begin{bmatrix} v_1 & v_2 & \dots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} v_1\lambda_1 & v_2\lambda_2 & \dots & v_n\lambda_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \tag{A-20}$$

We will define V as the matrix of column eigenvectors. We can rewrite the right side of Equation A-20 as the eigenvector matrix V multiplied by a diagonal matrix Λ that has the n eigenvalues along its diagonal, Equation A-21. After right multiplying V^{-1}, we derive the eigen decomposition of A, Equation A-24.

$$AV = \begin{bmatrix} v_1 & v_2 & \dots & v_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \dots \\ \dots & & \dots & \\ 0 & \dots & & \lambda_n \end{bmatrix} \tag{A-21}$$

$$AV = V\Lambda \tag{A-22}$$

$$AVV^{-1} = V\Lambda V^{-1} \tag{A-23}$$

$$A = V\Lambda V^{-1} \tag{A-24}$$

The eigen decomposition of A, Equation A-24, makes it very efficient to raise A to an integer exponent e. Consider first squaring A, Equation A-25.

$$\begin{aligned} A^2 &= AA \\ &= V\Lambda V^{-1}V\Lambda V^{-1} \\ &= V\Lambda\Lambda V^{-1} \\ &= V\Lambda^2 V^{-1} \end{aligned} \tag{A-25}$$

Equation A-26 for A to an integer exponent e is then derived by induction. This is very efficient to compute because raising Λ to the exponent e can be computed by simply raising the n diagonal elements to e.

$$A^e = V\Lambda^e V^{-1} \tag{A-26}$$

For the 2×2 matrix B, the eigen decomposition is the following:

$$B = V_2\Lambda_2 V_2^{-1} \tag{A-27}$$

$$V_2 = \begin{bmatrix} \frac{a-d-\sqrt{(a-d)^2+4bc}}{2c} & \frac{a-d+\sqrt{(a-d)^2+4bc}}{2c} \\ 1 & 1 \end{bmatrix} \tag{A-28}$$

$$\Lambda_2 = \begin{bmatrix} \frac{a+d-\sqrt{(a-d)^2+4bc}}{2\alpha} & 0 \\ 0 & \frac{a+d+\sqrt{(a-d)^2+4bc}}{2\alpha} \end{bmatrix} \tag{A-29}$$

$$V_2^{-1} = \frac{-c}{\sqrt{(a-d)^2+4bc}} \begin{bmatrix} 1 & -\frac{a-d+\sqrt{(a-d)^2+4bc}}{2c} \\ -1 & \frac{a-d-\sqrt{(a-d)^2+4bc}}{2c} \end{bmatrix} \tag{A-30}$$

$$\frac{\lambda_1+\lambda_2}{2} = \frac{a+d}{2\alpha} = \frac{Trace\,(A)}{2} \tag{A-31}$$

$$\lambda_1\lambda_2 = \frac{ad-bc}{\alpha} = Det\,(A) \tag{A-32}$$

We remember that the determinant of the multiplication of two matrices is the product of their determinants.

$$Det\,(AB) = Det\,(A)\,Det\,(B) \tag{A-33}$$

We can then prove that the determinant of a matrix is equal to the product of its eigenvalues.

$$A = V\Lambda V^{-1} \tag{A-34}$$

$$Det\,(A) = Det\left(V\Lambda V^{-1}\right) \tag{A-35}$$

$$= Det\,(V)\,Det\,(\Lambda)\,Det\left(V^{-1}\right) \tag{A-36}$$

$$= Det\left(V^{-1}V\Lambda\right) \tag{A-37}$$

$$= Det\,(\Lambda) \tag{A-38}$$

$$= \prod \lambda_i \tag{A-39}$$