The Lecture of January 25

We start by some examples. Of the following matrices, the left one is positive semidefinite.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The right hand side matrix is not p.s.d. since for $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ we have $x^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x = -2$. Further, of the following matrices

$$\begin{pmatrix} -1 & x \\ x & x \end{pmatrix} \begin{pmatrix} 0 & x \\ x & y \end{pmatrix}$$

the left one is not p.s.d. and the right one is, if

$$\begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} 0 & x \\ x & y \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 2cdx + d^2 y$$

is non-negative for all c, d iff $x = 0$ and $y \geq 0$. More generally

$$\begin{pmatrix} \epsilon & x \\ x & y \end{pmatrix}$$

is positive semidefinite, for all $\epsilon, y \geq 0$ and x satisfying $|x| \leq \sqrt{\epsilon y}$.

Recall the *Frobenius inner product* of two matrices $A \cdot B = \sum_i \sum_j A_{ij} B_{ij} = \text{trace}(A^T B)$. Further, we have

Proposition 1. *Let A, B be p.s.d., then* $\text{tr}(AB) \geq 0$.

A semidefinite program is given by

$$\begin{aligned} \text{minimize} \quad & c \cdot x && (1) \\ \text{subject to} \quad & x_1 A + \ldots + x_n A_n - B \succcurlyeq 0. && (2) \end{aligned}$$

Here, the A_i's and B are given symmetric matrices.

Just as with linear programs the dual of a semidefinite program can be seen as a valid lower bound of the primal. So, if in (2) we multiply both sides by a p.s.d. Y, we obtain, by Proposition 1,

$$(x_1 A + \ldots + x_n A_n - B) \cdot Y \geq 0.$$

Hence $x_1 A \cdot Y + \ldots + x_n A_n \cdot Y \geq B \cdot Y$ and we can achieve that $B \cdot Y \leq c \cdot x$, if we stipulate that

$$\begin{aligned} Y &\succcurlyeq 0 \\ A_1 \cdot Y &= c_1 \\ &\vdots \\ A_n \cdot Y &= c_n \end{aligned}$$

Then the above inequality turns into $c_1 x_1 + \ldots + c_n x_n \geq B \cdot Y$. Thus if we can find a Y subject to the above conditions so as to maximize $B \cdot Y$, then this maximum found is a lower bound for $c \cdot x$.

1 Strong Duality for SDPs

Denote by v_{primal} (v_{dual}, resp.) the supremum of the objective function of the primal (dual, resp.) SDP. A solution x to the primal program is called *strictly feasible*, if $x_1 A + \ldots + x_n A_n - B$ is positive definite.

We need the inhomogeneous version of the Farkas Lemma for SDPs.

Lemma 2. *Let $A_1, \ldots, A_n, B$ be symmetric. Then either there are $x_1 m, \ldots, x_n$ such that*

$$x_1 A_1 + \ldots + x_n A_n - B \succ 0$$

or there is a symmetric p.s.d $Y \neq 0$ such that

$$\begin{aligned} A_1 \cdot Y &= 0 \\ &\vdots \\ A_n \cdot Y &= 0 \\ B \cdot Y &\geq 0 \end{aligned}$$

Theorem 3 (Strong Duality). *If, the dual program has a feasible solution and the primal program has a strictly feasible solution, then $v_{primal} = v_{dual}$.*

Proof. We have already seen $v_{\text{primal}} \geq v_{\text{dual}}$. For the other direction, define matrices

$$A_i' = \begin{pmatrix} -c_i & 0 \\ 0 & A_i \end{pmatrix} \text{ and } B' = \begin{pmatrix} -v_{\text{primal}} & 0 \\ 0 & B \end{pmatrix}$$

Then the system $x_1 A_1' + \ldots + x_n A_n' - B' \succ 0$ has no solution in $x_1, \ldots, x_n$ as it stipulates that $v_{\text{primal}} > c \cdot x$ contrary to its definition. By Lemma 2 there is thus a p.s.d. $Y' \neq 0$ such that $A_i' \cdot Y' = 0$ and $B' \cdot Y' \succcurlyeq 0$. Note that Y' is of the form

$$Y' = \begin{pmatrix} y_{00} & y^T \\ y & Y \end{pmatrix}.$$

Hence $A_i' \cdot Y' = 0$ implies $A_i \cdot Y = y_{00} \cdot c_i$ and analogously, we have $B \cdot Y \geq y_{00} \cdot v_{\text{primal}}$.

Note that $y_{00} = 0$ would imply $Y \neq 0$, $A_i Y = 0$ and $B \cdot Y \geq 0$ whence by Lemma 2 unsolvability of the primal program would follow.

Therefore $y_{00} > 0$ and by scaling all entries of Y' we may assume $y_{00} = 1$. Consequently, Y is a solution of the dual problem with $B \cdot Y \geq v_{\text{primal}}$. □