The Lecture of January 25

We start by some examples. Of the following matrices, the left one is positive semidefinite.

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

The right hand side matrix is not p.s.d. since for \( x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) we have \( x^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x = -2 \). Further, of the following matrices

\[
\begin{pmatrix}
-1 & x \\
x & x
\end{pmatrix}
\begin{pmatrix}
0 & x \\
x & y
\end{pmatrix}
\]

the left one is not p.s.d. and the right one is, if

\[
\begin{pmatrix}
c & d \\
x & y
\end{pmatrix}
\begin{pmatrix}
c \\
d
\end{pmatrix} = 2cdx + d^2y
\]

is non-negative for all \( c, d \) iff \( x = 0 \) and \( y \geq 0 \). More generally

\[
\begin{pmatrix}
\epsilon & x \\
x & y
\end{pmatrix}
\]

is positive semidefinite, for all \( \epsilon, y \geq 0 \) and \( x \) satisfying \( |x| \leq \sqrt{\epsilon y} \).

Recall the Frobenius inner product of two matrices \( A \cdot B = \sum_i \sum_j A_{ij} B_{ij} = \text{trace}(A^T B) \). Further, we have

**Proposition 1.** *Let \( A, B \) be p.s.d., then \( \text{tr}(AB) \geq 0 \).*

A semidefinite program is given by

\[
\begin{align*}
\text{minimize} & \quad c \cdot x \\
\text{subject to} & \quad x_1 A + \ldots + x_n A_n - B \succeq 0.
\end{align*}
\]

Here, the \( A_i \)'s and \( B \) are given symmetric matrices.

Just as with linear programs the dual of a semidefinite program can be seen as a valid lower bound of the primal. So, if in (2) we multiply both sides by a p.s.d. \( Y \), we obtain, by Proposition 1,

\[
(x_1 A + \ldots + x_n A_n - B) \cdot Y \succeq 0.
\]
Hence \( x_1 A \cdot Y + \ldots + x_n A_n \cdot Y \geq B \cdot Y \) and we can achieve that \( B \cdot Y \leq c \cdot x \), if we stipulate that

\[
Y \succcurlyeq 0 \\
A_1 \cdot Y = c_1 \\
\vdots \\
A_n \cdot Y = c_n
\]

Then the above inequality turns into \( c_1 x_1 + \ldots + c_n x_n \geq B \cdot Y \). Thus if we can find a \( Y \) subject to the above conditions so as to maximize \( B \cdot Y \), then this maximum found is a lower bound for \( c \cdot x \).

1 **Strong Duality for SDPs**

Denote by \( v_{\text{primal}} \) (\( v_{\text{dual}} \), resp.) the supremum of the objective function of the primal (dual, resp.) SDP. A solution \( x \) to the primal program is called strictly feasible, if \( x_1 A + \ldots + x_n A_n - B \) is positive definite.

We need the inhomogeneous version of the Farkas Lemma for SDPs.

**Lemma 2.** Let \( A_1, \ldots, A_n, B \) be symmetric. Then either there are \( x_1 m, \ldots, x_n \) such that

\[
x_1 A_1 + \ldots + x_n A_n - B \succcurlyeq 0
\]

or there is a symmetric p.s.d \( Y \neq 0 \) such that

\[
A_1 \cdot Y = 0 \\
\vdots \\
A_n \cdot Y = 0 \\
B \cdot Y \geq 0
\]

**Theorem 3 (Strong Duality).** If, the dual program has a feasible solution and the primal program has a strictly feasible solution, then \( v_{\text{primal}} = v_{\text{dual}} \).

**Proof.** We have already seen \( v_{\text{primal}} \geq v_{\text{dual}} \). For the other direction, define matrices

\[
A'_i = \begin{pmatrix} -c_i & 0 \\
0 & A_i \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} -v_{\text{primal}} & 0 \\
0 & B \end{pmatrix}
\]

Then the system \( x_1 A'_1 + \ldots + x_n A'_n - B' \succcurlyeq 0 \) has no solution in \( x_1, \ldots, x_n \) as it stipulates that \( v_{\text{primal}} > c \cdot x \) contrary to its definition. By Lemma 2 there is thus a p.s.d \( Y' \neq 0 \) such that \( A'_i \cdot Y' = 0 \) and \( B' \cdot Y' \succcurlyeq 0 \). Note that \( Y' \) is of the form

\[
Y' = \begin{pmatrix} y_00 & y^T \\
y & Y \end{pmatrix}
\]

Hence \( A'_i \cdot Y' = 0 \) implies \( A_i \cdot Y = y_00 \cdot c_i \) and analogously, we have \( B \cdot Y \succeq y_00 \cdot v_{\text{primal}} \).

Note that \( y_00 = 0 \) would imply \( Y \neq 0 \), \( A_i Y = 0 \) and \( B \cdot Y \geq 0 \) whence by Lemma 2 unsolvability of the primal program would follow.

Therefore \( y_00 > 0 \) and by scaling all entries of \( Y' \) we may assume \( y_00 = 1 \). Consequently, \( Y \) is a solution of the dual problem with \( B \cdot Y \succeq v_{\text{primal}} \).