1 Overview

We will continue discussing the spectral approach. And we will hopefully finish up with a discussion of heuristics.

2 A spectral approach.

As mentioned last time, given a graph $G = (V, E)$, we consider

$$\min_x \frac{\sum_{e=u,v} ||x_u - x_v||^2}{\sum_{ij} ||x_i - x_j||^2}, \tag{1}$$

where x_i is an assignment for node i to a real number. That is, x is a vector with an element for each node in the graph. This has been called the Raleigh quotient.

As we discussed last time, the sparsity of the cut yields an upper bound on the value of this function. That is, address the following question.

Question 1: Assign all the nodes in S to x and all the nodes in $\overline{S}$ to y. Show that the resulting "raleigh" quotient functional has value at most sparsity of $(S, \overline{S})$.

We will consider regular graphs here. Last time, we mentioned a related problem, that is finding x such that

$$\max_{x \perp 1} \frac{||Ax||^2}{||x||^2} \tag{2}$$

The A above is the adjacency matrix of G (we leave the diagonal to diagonals and 1/d for each edge. We claimed the problem above could be solved by choosing a random vector that is orthogonal to the all ones and powering the matrix since for any vector

$$x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,$$

where v_i's are the eigenvectors of A (we note that we are being loose about the diagonal value of A) sorted by their eigenvalues.

Then, we have

$$A^t x = a_1 \lambda_1^t v_1 + a_2 \lambda_2^t v_2 + \cdots + a_n \lambda_n^t v_n,$$

After some period, the result becomes closer and closer to v_1. In the case, that we choose the diagonals to be d. The constant vector is v_1. (We assume the eigenvalues are positive.)

Thus, if x was orthogonal to v_1. Then the result converges to v_2, which happens to coincide with equation 2.

Now, we claim that that optimizing for either 1 or 2 yields the same vector.

2.1 Finding a sparse cut.

We find a vector x that optimizes the Raleigh quotient and compute the sparsity of all cuts of the form $(\{v_1, \ldots, v_i\}, \{v_{i+1}, \ldots, v_n\})$ where the v_i's are sorted with respect to x. We output the best such cut.

LEMMA 1
The sparsity of the cut that is found by the procedure above is at most $\Theta(\sqrt{V|E|}/N)$, *where* V *is*

$$\frac{\sum_{e=u,v}(x_u - x_v)^2}{\sum_{i,j}(x_i - x_j)^2}.$$

The proof of this statement is rather uninformative algebra. See, for example, Milena Mihail: Conductance and Convergence of Markov Chains-A Combinatorial Treatment of Expanders FOCS 1989: 526-531.

To get some idea of why it is true, consider a situation where each edge has the same length. Further, consider that the nodes are spaced on the line at equal intervals, say on the integers (for example, that $x_i = i$). Now each edge is of length $\Theta(N^2\sqrt{V}/\sqrt{|E|})$ along the line.

Question 2: Why?

Thus, a typical one of the $\Theta(N)$ cuts contains $\Theta(|E|N^2\sqrt{V}/sqrt|E|/N) = \Theta(\sqrt{|E|V}N)$ edges.

Furthermore, a cut in the middle third contains splits off $N/3$ nodes. Thus, the sparsity of this typical cut is at

$$\frac{\Theta(N\sqrt{V|E|})}{\Theta(N^2)} = \Theta(\sqrt{VE}/N).$$

To get some feeling for this approximation, we consider an expander graph. Here, the sparsity is $1/n$. Now, $V < 1/n$, thus we can find a cut of sparsity around $\Theta(1/n)$ assuming that the graph is bounded degree. This is thus a good bound for expanders.

For a line graph the sparsity is $1/n^2$, and $V < 1/n^2$. Thus, we can find a cut of sparsity $1/n^{3/2}$. That is, we get a factor of $\sqrt{n}$ approximation. This is less good, and the worst factor if we just use the theorem above. (though in fact, the cut algorithm using different methods can be shown to work quite well on this example.)