Lecture 12

1 Review

In the last lecture we discussed a class of language L called *Nick's Class*, or NC, such that $L \subseteq NC$ if $\exists$ a PRAM algorithm that runs in time $O(log^c n)$ and polynomial number of processors on an instance of length n. It was seen that the max-flow problem is generally **P**-complete. Today we look at perfect matching in bipartite graphs.

2 Perfect Matching

The problem is defined as follows: given a bipartite graph $G = (L, R, E)$, where $E \subseteq L \times R$, find $E' \subseteq E$ such that $\forall\ u \in L$ and $\forall\ v \in R$, $\exists\ e \in E'$ such that $(u, v) = e$. We say that such a matching is *perfect* if every vertex has exactly one edge incident on it.

A well known method of finding a perfect match if one exists is to reduce it to a max-flow problem by adding a source node s and a sink t, shown below in Figure 1. Assuming integral flows in the network, the maximum flow of the network then gives the size of the maximum matching. However, solving max-flow problems is not useful for parallel algorithms since we saw in the previous lecture that max-flow is **P**-complete.

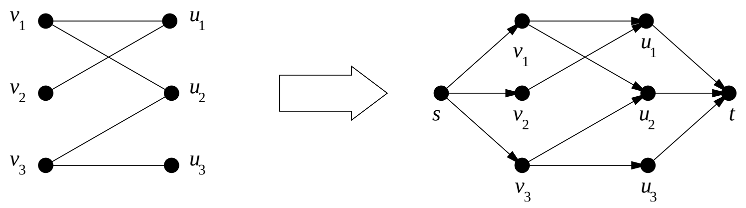

Figure 1: Reduction of perfect matching to max-flow.

2.1 Existence of Perfect Matching

Another approach is thus necessary, and we begin by showing how to proof the existence of a perfect matching. We represent the bipartite graph $G(L, R, E)$ as a matrix A of *indeterminants* x_{ij}s. Row i of A corresponds to a vertex $u_i \in L$, and column j corresponds to a vertex $v_j \in R$. Thus,

$$a_{ij} = \begin{cases} x_{ij} & \text{if } (u_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

This matrix is also known as the *Tutte matrix.* Using the graph in Figure 1 as an example, its representative matrix is

$$\begin{bmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & 0 \\ 0 & x_{32} & x_{33} \end{bmatrix} \tag{2}$$

We argue that a perfect matching exists if and only if the determinant of the representative matrix is non-zero. We make use of the Tutte polynomial like so:

$$\det(A) = \sum_{\pi} (-1)^{\sigma(\pi)} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \cdots a_{n\pi(n)} \tag{3}$$

where the determinant is the sum of products of permutations of a_{ij}s, with the first element obtained from the first row, the second element from the second row, and so on. Since a non-zero a_{ij} term signifies an existing edge between a vertex v and a vertex u, we have

$$\det(A) = \sum_{\pi \in P} (-1)^{\sigma(\pi)} x_{1\pi(1)} x_{2\pi(2)} x_{3\pi(3)} \cdots x_{n\pi(n)} \tag{4}$$

where P is the set of perfect matchings in G. Thus, a non-zero determinant means that there is a matching in G. Also, since the permutations are unique, the sum of non-zero permutations will also yield a non-zero result. This determinant should not be thought of as taking some numeric value, but as a function of the variables x_{ij}.

The problem of deciding if a perfect matching therefore reduces to the problem of determining if the Tutte matrix of the graph is non-singular. To reduce the complexity of this process, we make use of the following theorems:

Theorem 1
A Tutte polynomial is not a 0-polynomial if and only if there exists a perfect matching.

Theorem 2
Given a polynomial $p(x_1, x_2, \ldots, x_m)$ of degree d, if we choose $r_1, r_2, \ldots, r_m \in S$ and $p(\cdot) \neq 0$, then $\mathbf{Pr}[p(r_1, r_2, \ldots, r_m) = 0] \leq \frac{d}{|S|}$

Proof:

- Base case: A one-variable, nonzero polynomial of degree d has at most d zeros, and thus $\mathbf{Pr}[p(r_1) = 0] \leq \frac{d}{|S|}$ is always true.

- Inductive step: We rewrite $p(x_1, x_2, \ldots, x_m)$ as

$$p_d(x_1, \ldots, x_{m-1}) x_m^d + \ldots + p_{d-\alpha}(x_1, \ldots, x_{m-1}) x_m^{d-\alpha} + \ldots + p(x_1, \ldots, x_{m-1}) \tag{5}$$

  Focusing on $p_{d-\alpha}(x_1, \ldots, x_{m-1}) x_m^{d-\alpha}$, we note that the degree of $p_{d-\alpha}(x_1, \ldots, x_{m-1})$ is at most α. We assume that the theorem is correct for $p_{d-\alpha}(x_1, \ldots, x_{m-1})$, and we have

$$\mathbf{Pr}[p_{d-\alpha}(x_1, \ldots, x_{m-1}) = 0] \leq \frac{\alpha}{|S|} \tag{6}$$

From the base case, we also have

$$\mathbf{Pr}[x_m^{d-\alpha} = 0] \leq \frac{d-\alpha}{|S|} \tag{7}$$

Equations 6 and 7 completes the inductive step.

□

Computation of the determinants in parallel is accomplished using Chistov's algorithm, which is beyond the scope of this lecture. We next describe how to go about finding the matching.

2.2 Finding Matching

LEMMA 3
Isolation Lemma: Given a family $\mathcal{F}$ of subsets of $\{1,2,\ldots,m\}$, and weights uniformly chosen from $\{1,2,\ldots,2m\}$, the weight of the smallest weight subset is unique with probability at least $\frac{1}{2}$.

The proof is by Joel Spencer, and is given below.
PROOF: Let

$$\alpha(x) = \min_{E\in\mathcal{F}:x\notin E} w(E) - \min_{E\in\mathcal{F}:x\in E} w(E-x) \tag{8}$$

where $x \in \{1,2,\ldots,m\}$, E are subsets of $\{1,2,\ldots,m\}$, and $w(i)$ gives the sum of weights of subset i. Since there are $2m$ different weights each element can take on, we have

$$\mathbf{Pr}[w(x) = \alpha(x)] \leq \frac{1}{2m} \tag{9}$$

$$\text{or} \qquad \mathbf{Pr}[\exists x \text{ s.t. } w(x) = \alpha(x)] \leq \frac{1}{2} \qquad \text{(using union bound)} \tag{10}$$

Assume that there exists subsets A,$B \in \mathcal{F}$, both of minimum weight, i.e. $w(A) = w(B)$, and let $y \in A - B$. Then we have

$$\min_{E\in\mathcal{F}:y\notin E} w(E) = w(B) \tag{11}$$

$$\min_{E\in\mathcal{F}:y\in E} w(E-y) = w(A) - w(y) = w(B) - w(y) \tag{12}$$

$$\text{and thus} \qquad \alpha(y) = w(B) - (w(B) - w(y)) = w(y) \tag{13}$$

Equation 13 occurs with probability $\leq \frac{1}{2}$, completing the proof. □

Since a union bound is used, the bound can be made arbitrarily small by choosing weights from a larger set of numbers.

2.3 Application

We now elaborate on a specific method to find a matching. Let $\mathcal{F}$ be the set of perfect matchings, and we assign random weights $w_{ij} \in \{1, 2, \ldots, 2n^2(n^2)\}$. Letting $B = A$, and replacing x_{ij} with $2^{w_{ij}}$, from the Isolation Lemma, there exists a unique minimum weight matching with probability $\geq 1 - \frac{1}{2n^2}$.

Then, computing $\det(B)$ using Chistov's algorithm, we have

$$\det(B) = \sum_{\pi} (-1)^{\sigma(\pi)} 2^{\sum_i w_{i\pi(i)}} \tag{14}$$

The value of the summation can be viewed as a bit string of ones and zeros, where the term $2^{\sum_i w_{i\pi(i)}}$ contributes a bit at the position $\sum_i w_{i\pi(i)}$. If we assume that the matching with minimal weight is unique, then the least significant 1 in this bit string gives the weight of the matching with minimal weight. This is because each summand that contributes a 1 is a non-zero term in the determinant which represents a matching, and if the minimal is unique, then there cannot be any term that results in a 1 at the same position or at a more significant position.

Next, for each x_{ij}, we compute $\det(B_{ij})$ where B_{ij} is B with the ith row and the jth column removed. If x_{ij} was in the minimal weight matching, then $\det(B_{ij})$ should yield an even smaller minimal weight (lowest significant 1-bit). In particular, the π term that yielded the least significant 1-bit in $\det(B)$ is

$$(-1)^{\sigma(\pi)} 2^{(\sum_i w_{i\pi(i)}) - w_{ij}} \tag{15}$$

Thus, if the positions of least significant 1-bits of $\det(B)$ and $\det(B_{ij})$ differ by w_{ij}, then x_{ij} is in the matching high probability. Using union bound, we can find edges in the matching with probability $\geq \frac{1}{2}$.