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27.1 Auction

27.1.1 Review

Last time we discussed about quasi-linear utilities.

$$K = K_0 \times \mathbf{R}^n \tag{27.1}$$

Where K_0 is the core outcome and $\mathbf{R}^n$ is the payment to players. The utility and payment functions are

$$u_i(k_0, (x_1, .. , x_n)) = u_{i0}(k_0) + x_i \tag{27.2}$$

$$P[u_i,..,u_n] = \arg\max_k \sum_i u_{i0}(k_0) \tag{27.3}$$

Where u_i is the utility an outcome to the ith player, u_{i0} is the utility of the outcome before ith player joins, k_0 is an outcome, x_i is the payment of each player. The utility of an outcome to a player i is the utility before he/she joins plus the payment he/she pays. And the payment of all the utilities is the maximum of the sum of all utilities out of all possible outcomes.

Auction is basically money transfer and we want to have the bidders give their true value bid. There are three examples we saw last time: Vickery's Auction, Public Good, and Shortest Path Auction. Vickery's Auction ensures the bidder will tell the truth by making the winner pay the second highest bid. This way if a bidder bid higher than his true value, he might end up paying more. If he bids lower, he might not get the bid. Public Good problems are similar. The person only pays up to the difference between the cost of the good and the total sum of everyone else's payments. For Shortest Path problems, again they are similar. There are four properties:

1. Monotone (by increasing your bid you don't get included)
2. Nobody gets < bid
3. No payment to edges not selected
4. Truthful (truth telling is the dominant strategy)

If these conditions are satisfied, they are solutions defined by Vickery-Clarke-Groves (VCG).

27.1.2 Proof of cost

Claim:

Any protocol satisfying 1-4 from above must pay *shortest path* + $\frac{n}{2}$ *D path* (Fig. 27.1)

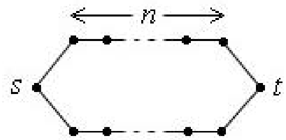

Figure 27.1: Example of a shortest path auction

Proof:

(1)-(4) implies protocol is: get bids, select path, pay everybody selected his/her threshold bid. Threshold bid is defined as the player's bid plus the difference between the shortest path and the next shortest path.

For a graph with two paths differing only *d* (Fig. 27.2)

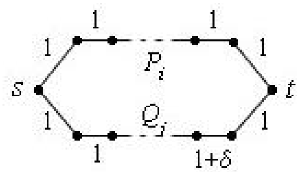

Figure 27.2: Lowest cost case: two paths with small difference

It can be transformed into a bipartite graph (Fig. 27.3) by comparing the edges and making a directed edge for each comparison

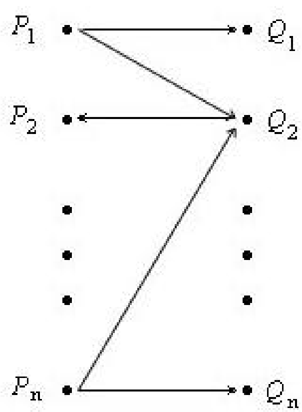

Figure 27.3: Directed bipartite graph representing edge comparison of Fig. 27.2

Since every edge is compared to another edge

$$\Sigma \; out \; degrees = n^2 \tag{27.4}$$

at least one node has

$$out \; degree \geq \frac{n}{2} \tag{27.5}$$

then each lower bid edge has this minimum threshold

$$threshold \geq 1 + d \tag{27.6}$$

Therefore summing over all edges

$$Payment \geq shortest\ path\ (n) + \frac{n}{2} * D\ path\ (d) \qquad (27.7)$$

For more details, refer to [ESS03].

27.2 Coalition Games

27.2.1 Definition

n players: $[1, \ldots, n]$
values: $2^{[n]} \rightarrow \boldsymbol{R}_{+}$

One interpretation:

$$S \subseteq [n] \qquad (27.8)$$

$$v(s) = \max_{a_S} \min_{a_{\bar{S}}} \sum_{j \in S} P_j (a_S, a_{\bar{S}}) \qquad (27.9)$$

$$\sum_{i=1}^{n} x_i = v([n]) \qquad (27.10)$$

S is the set of players, $v(S)$ is the value of that set, and x_i is the allocation (payment) of each player. In this interpretation, the sum of the allocation values to all the players is the value of the solution.

27.2.2 Solution

A solution concept is a set of allocations that results in non-empty sets.

SC1: The core

$$\forall S \quad \sum_{i \in S} x_i \geq v[S] \qquad (27.11)$$

the sum of all the players' payments is equal to or more than the value of the game.

One example is the 3-majority game, for example:

$$\begin{gathered} v(1, 2, 3) = 1 \\ v(2, 3) = v(1, 3) = v(1, 2) = \alpha \\ v(1) = v(2) = v(3) = v(\text{Æ}) = 0 \end{gathered} \qquad (27.12)$$

using the core's solution

$$\begin{gathered} x_1 + x_2 + x_3 \geq 1 \\ x_1 + x_2 \geq \alpha \\ x_2 + x_3 \geq \alpha \\ x_3 + x_1 \geq \alpha \end{gathered} \qquad (27.13)$$

Adding up the last 3 equations and setting $x_1 + x_2 + x_3$ to the minimum 1 results $2 \geq 3\alpha$. If $\alpha > \frac{2}{3}$, there would be no way to split 1 for x_1, x_2, x_3 and make them play together because either pair of them can make more money. So no one will play. If $\alpha < \frac{2}{3}$, then no set of players should have the incentive to leave the game and do something else to get more payment and the solution for the core ≠ Æ.

$$v(S) = \left\lfloor \frac{|S|}{2} \right\rfloor \tag{27.14}$$

Theorem: if v is convex

$$(v(S) + v(T) \leq v(S \cup T) + v(S \cap T)) \tag{27.15}$$

then core ≠ Æ, $x_i = v(\{1, .., l\}) - v(\{1, .., l\text{-}1\})$

Another example is the voting game:

$v[S] = 0$ or 1, 1 implies $i \in S$, players can be yes/no/veto.

Theorem: if there are no veto players then the core solution is $\varnothing$

$$\text{core} = \{x : x_i = 0 \text{ if } i \text{ is not a veto player}\} \tag{27.16}$$

27.2.3 Shapley Value

$$x_i = \mathrm{E}_p\,[v(\{j{:}p(j) \leq p(i)\}) - v(\{j{:}p(j) < p(i)\})] \tag{27.17}$$

E_p is the expectation average over all p, $p(i)$ is the arrival order of players, and $[\, v(\{j{:}p(j) \leq p(i)\}) - v(\{j{:}p(j) < p(i)\})$ is the difference i made by arriving.

e.g. UN Security Council problem

$$v[S] = 1 \text{ if } S \geq \{1,.., 5\},\ |S| \geq 8 \tag{27.18}$$

$v[S] = 1$ means a vote gets passed.

Shapley Theorem: The Shapley value is the only allocation rule that satisfies Shapley's Axioms.

Shapley's Axioms:

Symmetry: $$v(S + i) = v(S + j)\ \ \forall\ i, j \notin S \Rightarrow x_i^v = x_j^v \tag{27.19}$$

Dummy: $$v(S + i) = v(S) + v(\{i\})\ \ \forall\ i \notin S,\ x_i^v = v(\{i\}) \tag{27.20}$$

Additive: $$x_i^{v+v'} = x_i^v + x_i^{v'} \tag{27.21}$$

Reference:

[ESS03] E. Elkind and A. Sahai and K. Steiglitz, "Frugality in Path Auctions", Preliminary Version, April 2003. http://www.cs.princeton.edu/~elkind/frugal.ps.