The multiplicative weights framework.

Experts framework.

n experts.

Experts framework.

n experts.

Every day, each offers a prediction.

Experts framework.

n experts.

Every day, each offers a prediction.

“Rain” or “Shine.”

Experts framework.

n experts.

Every day, each offers a prediction.

“Rain” or “Shine.”

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1				$\cdots$	
Expert 2				$\cdots$	
Expert 3				$\cdots$	
$\vdots$				$\cdots$	

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine			$\cdots$	
Expert 2	Shine			$\cdots$	
Expert 3	Rain			$\cdots$	
$\vdots$	$\vdots$			$\cdots$	

Rained!

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain		$\cdots$	
Expert 2	Shine	Shine		$\cdots$	
Expert 3	Rain	Rain		$\cdots$	
$\vdots$	$\vdots$	$\vdots$		$\cdots$	

Rained! Shined!

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined!

Experts framework.

n experts.

Every day, each offers a prediction.

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	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Experts framework.

n experts.

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	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Sort of.

Experts framework.

n experts.

Every day, each offers a prediction.

"Rain" or "Shine."

	Day 1	Day 2	Day 3	$\cdots$	Day T
Expert 1	Shine	Rain	Shine	$\cdots$	
Expert 2	Shine	Shine	Shine	$\cdots$	
Expert 3	Rain	Rain	Rain	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	Shine	$\cdots$	

Rained! Shined! Shined! $\cdots$

Whose advice do you follow?

"The one who is correct most often."

Sort of.

How well do you do?

Infallible expert.

One of the experts is infallible!

Infallible expert.

One of the experts is infallible!

Your strategy?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never!

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make?

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

Adversary designs setup to watch who you choose, and make that expert make a mistake.

Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

(A) 1

(B) 2

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Adversary designs setup to watch who you choose, and make that expert make a mistake.

$n-1$!

Concept Alert.

Note.

Concept Alert.

Note.

Adversary:

Concept Alert.

Note.

Adversary:
makes you want to look bad.

Concept Alert.

Note.

Adversary:
makes you want to look bad.
”You could have done so well”...

Concept Alert.

Note.

Adversary:
makes you want to look bad.
”You could have done so well”...
but you didn’t!

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Adversary:
makes you want to look bad.
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Concept Alert.

Note.

Adversary:
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Concept Alert.

Note.

Adversary:
  makes you want to look bad.
  ”You could have done so well”...
    but you didn’t! ha..ha!

Analysis of Algorithms: do as well as possible!

Back to mistake bound.

Infallible Experts.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$

Lower bound: adversary argument.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
  Lower bound: adversary argument.
  Upper bound:

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
Lower bound: adversary argument.
Upper bound: every mistake finds fallible expert.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
  Lower bound: adversary argument.
  Upper bound: every mistake finds fallible expert.

Better Algorithm?

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
  Lower bound: adversary argument.
  Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

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Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.

Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n-1$
Lower bound: adversary argument.
Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.

What you would do anyway!

Alg 2: find majority of the perfect

How many mistakes could you make?

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n - 1$

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n - 1$

At most $\log n$!

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n-1$

At most $\log n$!

When alg makes a *mistake*,
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mistake $\rightarrow$ $\leq n/2$ perfect experts
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$\vdots$

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

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At most $\log n$!

When alg makes a *mistake*,
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Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts

$\vdots$

mistake $\rightarrow$ ≤ 1 perfect expert

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

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At most $\log n$!

When alg makes a *mistake*,
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mistake $\rightarrow$ $\leq n/2$ perfect experts
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How many mistakes could you make?

(A) 1

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At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
$\vdots$
mistake $\rightarrow$ ≤ 1 perfect expert

≥ 1 perfect expert

Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1

(B) 2

(C) $\log n$

(D) $n - 1$

At most $\log n$!

When alg makes a *mistake*,
$|$"perfect" experts$|$ drops by a factor of two.

Initially n perfect experts
mistake $\rightarrow$ $\leq n/2$ perfect experts
mistake $\rightarrow$ $\leq n/4$ perfect experts
$\vdots$
mistake $\rightarrow$ ≤ 1 perfect expert

≥ 1 perfect expert $\rightarrow$ at most $\log n$ mistakes!

Imperfect Experts

Goal?

Imperfect Experts

Goal?

Do as well as the best expert!

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Imperfect Experts

Goal?

Do as well as the best expert!

Algorithm. Suggestions?

Go with majority?

Penalize inaccurate experts?

Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Analysis: weighted majority

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function:

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

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Each mistake:

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2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2?

1. Initially: $w_i = 1$.
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Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2? factor of $\frac{1}{2}$?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:

total weight of incorrect experts reduced by

-1? -2? factor of $\frac{1}{2}$?

each incorrect expert weight multiplied by $\frac{1}{2}$!

1. Initially: $w_i = 1$.
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Goal: Best expert makes m mistakes.

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For best expert, b, $w_b \geq \frac{1}{2^m}$.

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-1? -2? factor of $\frac{1}{2}$?

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total weight decreases by

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:

total weight of incorrect experts reduced by

-1? -2? factor of $\frac{1}{2}$?

each incorrect expert weight multiplied by $\frac{1}{2}$!

total weight decreases by

factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \to w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2? factor of $\frac{1}{2}$?
each incorrect expert weight multiplied by $\frac{1}{2}$!
total weight decreases by
factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?
mistake $\rightarrow \geq$ half weight with incorrect experts
($\geq \frac{1}{2}$ total.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: weighted majority

Goal: Best expert makes m mistakes.

Potential function: $\sum_i w_i$. Initially n.

For best expert, b, $w_b \geq \frac{1}{2^m}$.

Each mistake:
total weight of incorrect experts reduced by
-1? -2? factor of $\frac{1}{2}$?
each incorrect expert weight multiplied by $\frac{1}{2}$!
total weight decreases by
factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?
mistake $\rightarrow \geq$ half weight with incorrect experts
($\geq \frac{1}{2}$ total.

Mistake $\rightarrow$ potential function decreased by $\frac{3}{4}$.
$\implies$ for M is number of mistakes that:

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

Analysis: continued.

$$\frac{1}{2^m} \le \sum_i w_i \le \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \le \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \le -M \log(4/3) + \log n.$$

Analysis: continued.

$$\frac{1}{2^m} \le \sum_i w_i \le \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \le \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \le -M \log(4/3) + \log n.$$

Solve for M.

$$M \le (m + \log n) / \log(4/3)$$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \leq -M \log(4/3) + \log n.$$

Solve for M.

$M \leq (m + \log n) / \log(4/3) \leq 2.4(m + \log n)$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \leq -M \log(4/3) + \log n.$$

Solve for M.

$$M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n)$$

Multiple by $1-\varepsilon$ for incorrect experts...

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \leq -M \log(4/3) + \log n.$$

Solve for M.

$$M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n)$$

Multiple by $1 - \varepsilon$ for incorrect experts...

$$(1-\varepsilon)^m \leq \left(1 - \frac{\varepsilon}{2}\right)^M n.$$

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

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Solve for M.

$$M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n)$$

Multiple by $1 - \varepsilon$ for incorrect experts...

$$(1-\varepsilon)^m \leq \left(1 - \frac{\varepsilon}{2}\right)^M n.$$

Massage...

Analysis: continued.

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \leq -M \log(4/3) + \log n.$$

Solve for M.

$$M \leq (m + \log n) / \log(4/3) \leq 2.4(m + \log n)$$

Multiple by $1 - \varepsilon$ for incorrect experts...

$$(1-\varepsilon)^m \leq \left(1 - \frac{\varepsilon}{2}\right)^M n.$$

Massage...

$$M \leq 2(1+\varepsilon)m + \frac{2 \ln n}{\varepsilon}$$

Analysis: continued.

$$\frac{1}{2^m} \le \sum_i w_i \le \left(\frac{3}{4}\right)^M n.$$

m - best expert mistakes M algorithm mistakes.

$$\frac{1}{2^m} \le \left(\frac{3}{4}\right)^M n.$$

Take log of both sides.

$$-m \le -M \log(4/3) + \log n.$$

Solve for M.

$$M \le (m + \log n)/\log(4/3) \le 2.4(m + \log n)$$

Multiple by $1-\varepsilon$ for incorrect experts...

$$(1-\varepsilon)^m \le \left(1 - \frac{\varepsilon}{2}\right)^M n.$$

Massage...

$$M \le 2(1+\varepsilon)m + \frac{2\ln n}{\varepsilon}$$

Approaches a factor of two of best expert performance!

Best Analysis?

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Consider two experts: A,B

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Which is worse?

(A) A correct even days, B correct odd days

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Factor of (almost) two worse!

Randomization

Better approach?

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Use?

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Proof Idea: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

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No factor of 2 loss!

Gains.

Why so negative?

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where G^* is payoff of best expert.

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Multiplicative weights framework!

Applications next!

Strategic Games.

N players.

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Each player has strategy set. $\{S_1, \ldots, S_N\}$.

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Example:

2 players

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Each player has strategy set. $\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \Re^N$).

Example:

2 players

Player 1: $\{$ **D**efect, **C**ooperate $\}$.
Player 2: $\{$ **D**efect, **C**ooperate $\}$.

Strategic Games.

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Player 2: $\{$ **D**efect, **C**ooperate $\}$.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

	C	D
C	(3,3)	(0,5)
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What is the best thing for the players to do?

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Nash Equilibrium: neither player has incentive to change strategy.

Digression..

What situations?

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Prisoner's dilemma:

Digression..

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Two prisoners separated by jailors and asked to betray partner.

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This class(today): simpler version.

Two Person Zero Sum Games

2 players.

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Each player has strategy set:
m strategies for player 1 n strategies for player 2

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Mixed Strategies.

	R	P	S
R	0	1	-1
P	-1	0	1
S	1	-1	0

How do you play?

Mixed Strategies.

		R	P	S
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

How do you play?

Player 1: play each strategy with equal probability.

Mixed Strategies.

		R	P	S
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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Mixed Strategies.

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		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Definitions.

Mixed strategies: Each player plays distribution over strategies.

Mixed Strategies.

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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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Payoffs?

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

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Payoffs? Can't just look it up in matrix!.

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Average Payoff.

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Average Payoff. Expected Payoff.

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Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$

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Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

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Payoffs: Equilibrium.

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$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

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Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

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Payoffs: Equilibrium.

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$$E[X] = 0.$$

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Payoffs: Equilibrium.

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$$E[X] = 0.^{1}$$

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Equilibrium

		R	P	S
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Will Player 1 change strategy?

Equilibrium

		R	P	S
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Will Player 1 change strategy? Mixed strategies uncountable!

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Expected payoffs for pure strategies for player 1.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock?

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		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$

Equilibrium

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		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper?

Equilibrium

		R	P	S
		$.3\bar{3}$	$.3\bar{3}$	$.3\bar{3}$
R	$.3\bar{3}$	0	1	-1
P	$.3\bar{3}$	-1	0	1
S	$.3\bar{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Equilibrium

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No better pure strategy.

Equilibrium

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Mixed strat. payoff is weighted av. of payoffs of pure strats.

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$E[X]=\sum_{(i,j)}(Pr[i]\times Pr[j])X(i,j)$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Mixed strategy can't be better than the best pure strategy.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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P	$.3\overline{3}$	-1	0	1
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		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Equilibrium!

Another example plus notation.

Rock, Paper, Scissors, prEempt.

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PreEmpt ties preEmpt, beats everything else.

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Payoffs.

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	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium?

Another example plus notation.

Rock, Paper, Scissors, prEempt.
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	R	P	S	E
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Equilibrium? **(E,E)**.

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Rock, Paper, Scissors, prEempt.
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	R	P	S	E
R	0	1	-1	1
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Equilibrium? **(E,E)**. Pure strategy equilibrium.

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Notation:

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Payoffs.

	R	P	S	E
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Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
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	R	P	S	E
R	0	1	-1	1
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Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Playing the boss...

Row has extra strategy:Cheat.

Playing the boss...

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Ties with Rock, Paper, beats scissors.

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Row has extra strategy:Cheat.
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Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

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Note: column knows row cheats.

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Why play?
Row is column's advisor.

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Equilibrium: play the boss...

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Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$.

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Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

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Payoff? Remember: weighted average of pure strategies.

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Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1$

Equilibrium: play the boss...

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Payoff? Remember: weighted average of pure strategies.

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Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$

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Strategy 4: $\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1$

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Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

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Why not play just one? Change payoff for other player!

Two person zero sum games.

$m \times n$ payoff matrix A.

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$$p(x, y) = x^t A y$$

That is,

$$\sum_i x_i \left(\sum_j a_{i,j} y_j \right) = \sum_j \left(\sum_i x_i a_{i,j} \right) y_j.$$

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(No better column strategy, no better row strategy.)

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Best Response

Column goes first:

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Find y, where best row is not too low..

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Example: Roshambo.

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Example: Roshambo. Value of R?

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Row goes first:
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Best Response

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Duality.

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Doesn't matter who plays first!

Equilibrium existence.

Linear programs.

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Column player: find y to maximize row payoffs.

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Approximate Equilibrium: $C(x) - R(y) \leq \varepsilon$.

Aproximate equilibrium ...

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Always: $R(y) \leq C(x)$
  For $R(y)$, minimizer x "goes second", but goes first for $C(x)$.

Strategy pair: (x, y)

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$T = \frac{\ln n}{\varepsilon^2}$, $R(y^*) \leq 1$

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Left as exercise.

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More comments

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More comments

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Versus Linear Programming: $O(n^3 m)$

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More comments

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$T = \frac{\ln n}{\varepsilon^2} \to O(nm\frac{\log n}{\varepsilon^2})$. Basically linear!

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Various assumptions: $[0,1]$ losses, other ranges takes some work.

Toll/Congestion

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Given $(s_1, t_1) \ldots (s_k, t_k)$.
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Column: choose edge.
Row pays if column chooses edge on any path.

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Runtime only dependent on m and T (number of days.)

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Will use gain and $[0, \rho]$ version of experts:

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$\square$

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Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)

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Learning just a bit.

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Example: set of labelled points, find hyperplane that separates.

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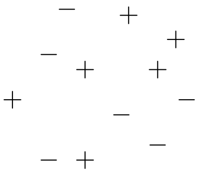

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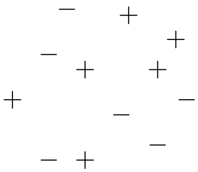

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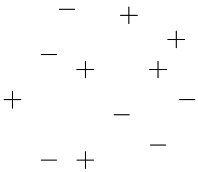

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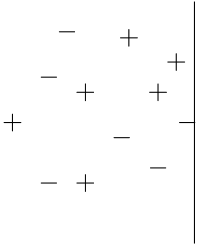

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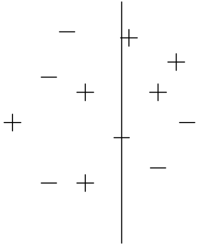

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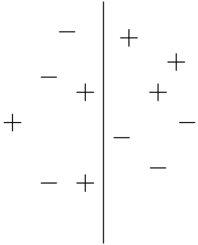

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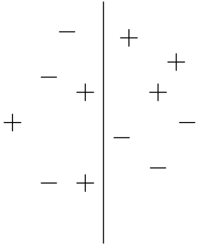

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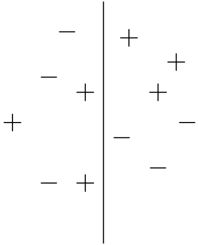

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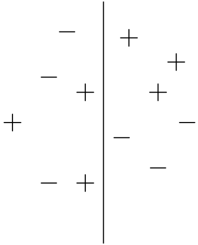

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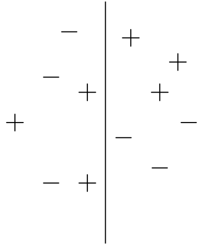

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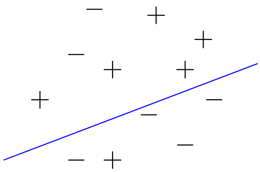

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That's a really strong learner!

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Can one use weak learning to produce strong learner?

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Boosting: use a weak learner to produce strong learner.

Poll.

Given a weak learning method (produce ok hypotheses.)

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(B) No

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The idea: Multiplicative Weights.

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The idea: Multiplicative Weights.
Standard online optimization method reinvented in many areas.

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Really? Proof?

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Second implies: $(1-\varepsilon)^x \leq e^{-\varepsilon x}$, by exponentiation.

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Set $\varepsilon = \gamma$, take logs.

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