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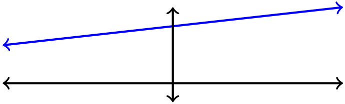

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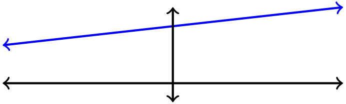

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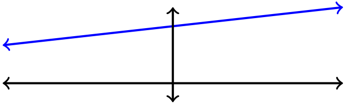

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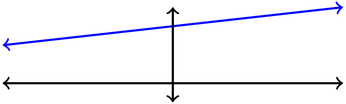

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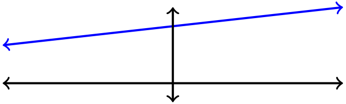

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Constrained optimization: calculus on an interval.

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There are no "vertices" in the "feasible region."

Convex hyperplane separator.

If a point $b \notin P$, for a set P which is convex.

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then there is y, s.t., $y^T x > y^T b, \forall x \in P$.

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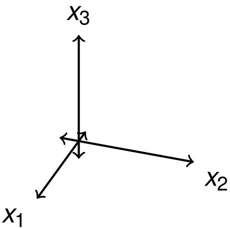

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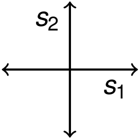

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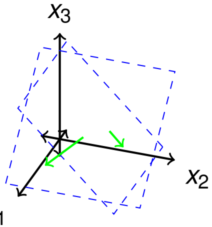

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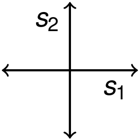

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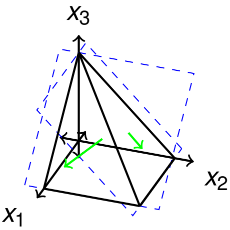

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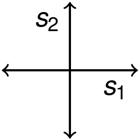

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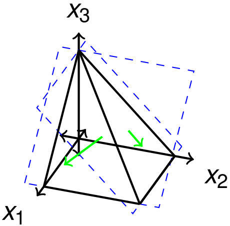

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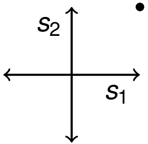

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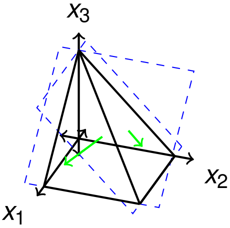

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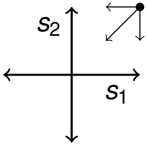

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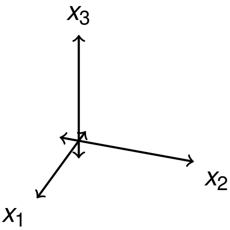

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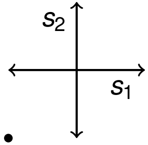

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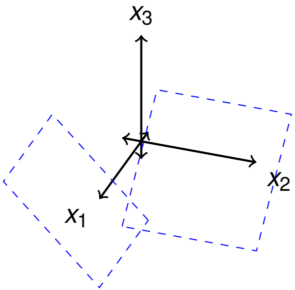

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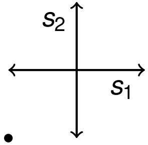

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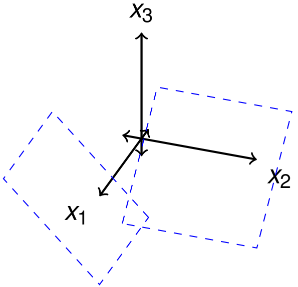

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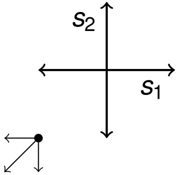

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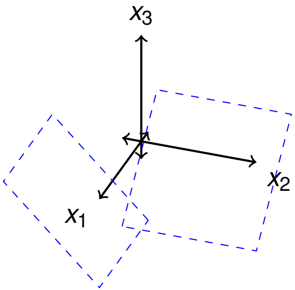

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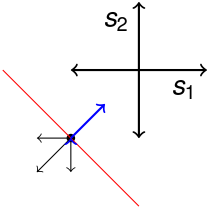

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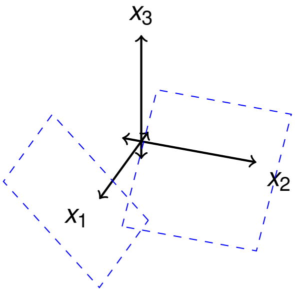

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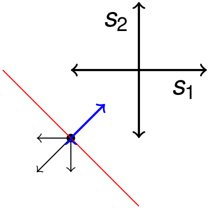

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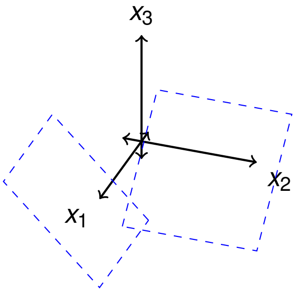

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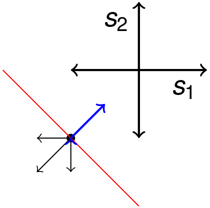

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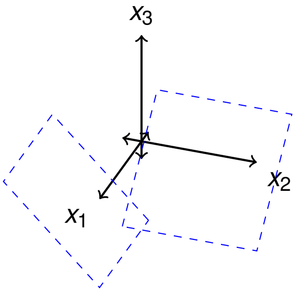

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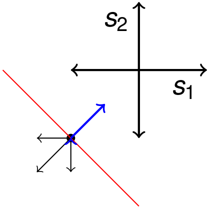

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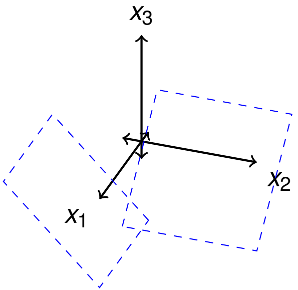

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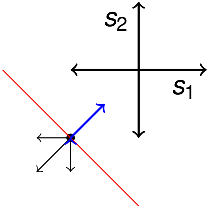

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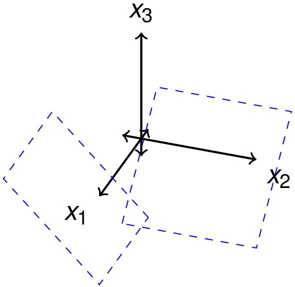

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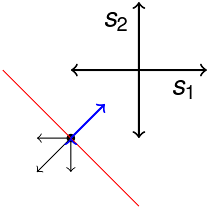

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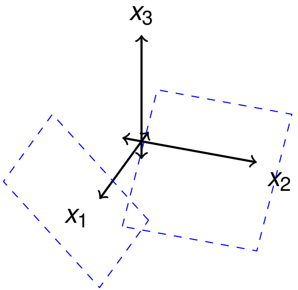

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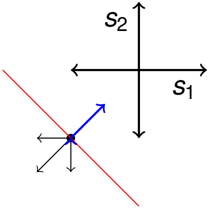

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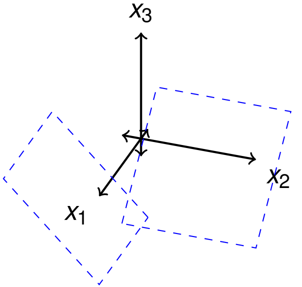

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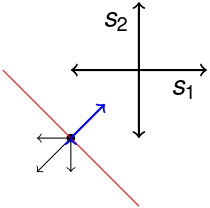

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Strong Duality

(From Goemans notes.)

Primal P

$$z^* = \min c^T x$$
$$Ax = b$$
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$$\begin{aligned} x^T c - b^T y &= x^T c - x^T A^T y \\ &= x^T (c - A^T y) \\ &\geq 0 \end{aligned}$$

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$$\begin{array}{ll} \min & f(x) \\ \text{subject to } f_i(x) \le 0, & i = 1, ..., m \end{array}$$

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$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$

If (primal) x value v

For all $\lambda \geq 0$ with $L(x, \lambda) \leq v$

Maximizing: λ only positive when? $f_i(x) = 0$.

Lagrangian:constrained optimization.

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to } f_i(x) \leq 0, \quad & i = 1, ..., m \end{aligned}$$

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Optimum value of program is at least α

Lagrangian:constrained optimization.

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If (primal) x value v
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Optimum value of program is at least α

Primal problem:
x, that minimizes $L(x, \lambda)$ over all $\lambda \ge 0$.

Lagrangian:constrained optimization.

$$\begin{array}{ll} \min & f(x) \\ \text{subject to } f_i(x) \leq 0, & i = 1, ..., m \end{array}$$

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If there is λ with $L(x,\lambda) \geq \alpha$ for all x

Optimum value of program is at least α

Primal problem:

x, that minimizes $L(x,\lambda)$ over all $\lambda \geq 0$.

Dual problem:

λ, that maximizes $L(x,\lambda)$ over all x.

Linear Program.

$\min cx, Ax \geq b$

Linear Program.

$\min cx, Ax \geq b$

$$\begin{aligned} \min \quad & c \cdot x \\ \text{subject to } & b_i - a_i \cdot x \leq 0, \qquad i = 1, ..., m \end{aligned}$$

Linear Program.

$\min cx, Ax \geq b$

$$\begin{aligned} \min \quad & c \cdot x \\ \text{subject to } & b_i - a_i \cdot x \leq 0, \qquad i = 1, ..., m \end{aligned}$$

Lagrangian (Dual):

Linear Program.

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Lagrangian (Dual):

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Lagrangian (Dual):

$L(\lambda, x) = cx + \sum_i \lambda_i (b_i - a_i x_i)$.

or

$L(\lambda, x) = -(\sum_j x_j (a_j \lambda - c_j)) + b\lambda$.

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Best λ?

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Best λ? Good against every x?

Linear Program.

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Best λ? Good against every x? Any term $(a_j \lambda - c_j) \neq 0$ is bad.

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$\max b \cdot \lambda$

Linear Program.

$\min cx, Ax \geq b$

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Best λ? Good against every x? Any term $(a_j\lambda - c_j) \neq 0$ is bad.
$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

Linear Program.

$\min cx, Ax \geq b$

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Best λ? Good against every x? Any term $(a_j \lambda - c_j) \neq 0$ is bad.

$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

Why is this good?

Linear Program.

$\min cx, Ax \geq b$

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Best λ? Good against every x? Any term $(a_j \lambda - c_j) \neq 0$ is bad.

$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

Why is this good? Every x is the same.

Linear Program.

$\min cx, Ax \geq b$

$$\begin{aligned} \min \quad & c \cdot x \\ \text{subject to } b_i - a_i \cdot x \leq 0, \quad & i = 1, ..., m \end{aligned}$$

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$\max b\lambda$,

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Best λ? Good against every x? Any term $(a_j\lambda - c_j) \neq 0$ is bad.

$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

Why is this good? Every x is the same.

$\max b\lambda, \lambda^T A = c,$

Linear Program.

$\min cx, Ax \geq b$

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Why is this good? Every x is the same.

$\max b\lambda, \lambda^T A = c, \lambda \geq 0$

Linear Program.

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$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

Why is this good? Every x is the same.

$\max b\lambda, \lambda^T A = c, \lambda \geq 0$

Dual to linear program.

Interior point on the central path.

Find x, that minimizes $f_0(x)$ subject to

Interior point on the central path.

Find x, that minimizes $f_0(x)$ subject to

$f_i(x) \leq 0, i = 1, \ldots m.$

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Find x, that minimizes $f_0(x)$ subject to

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Central path:

Interior point on the central path.

Find x, that minimizes $f_0(x)$ subject to

$f_i(x) \leq 0, i = 1, \ldots m.$

Central path:

$\min tf_0(x) - \sum_{i=1} m \ln(-f_i(x))$

Interior point on the central path.

Find x, that minimizes $f_0(x)$ subject to

$f_i(x) \leq 0, i = 1, \ldots m.$

Central path:

$\min t f_0(x) - \sum_{i=1} m \ln(-f_i(x))$

The minimizer, $x(t)$, form the **central path.**

Interior point on the central path.

Find x, that minimizes $f_0(x)$ subject to

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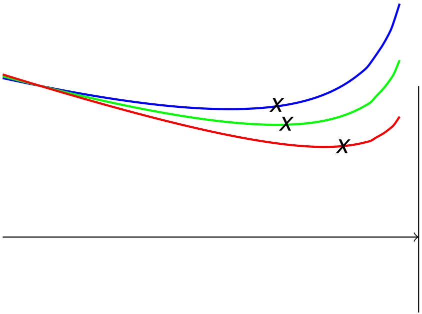
The minimizer, $x(t)$, form the **central path.**



The sequence of x's are "central path".

Lagrangian Dual and Central Path.

$$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$$

Lagrangian Dual and Central Path.

$$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$$

Optimality condition?

Lagrangian Dual and Central Path.

$$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$$

Optimality condition?

Lagrangian Dual and Central Path.

$$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$$

Optimality condition?

Derivative:

Lagrangian Dual and Central Path.

$\min t f_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

Or, $\nabla f_0(x) = -\sum_{i=1} \frac{\nabla f_i(x)}{tf_i(x)}$

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Lagrangian Dual and Central Path.

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Recall, Lagrangian: $L(\lambda, x) = f_0(x) + \sum_i \lambda_i f_i(x)$.

Fix λ, optimize for x^* give valid lower bound on solution.

Lagrangian Dual and Central Path.

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Optimality Condition.

Lagrangian Dual and Central Path.

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Lagrangian Dual and Central Path.

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Fix λ, optimize for x^* give valid lower bound on solution.

Optimality Condition.

Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0$.

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

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Fix λ, optimize for x^* give valid lower bound on solution.
Optimality Condition.
Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0$.

Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$.

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

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Fix λ, optimize for x^* give valid lower bound on solution.

Optimality Condition.

Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0$.

Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$!

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$ $\nabla f_0(x) + \sum_{i=1} \frac{1}{tf_i(x)} \nabla f_i(x) = 0$

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Fix λ, optimize for x^* give valid lower bound on solution.

Optimality Condition.

Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0$.

Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$! Same optimal point!

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

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Value?

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

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Optimality Condition.
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Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$! Same optimal point!

Value? $f_0(x) + \sum_{i=1} \lambda_i f_i(x) = f_0(x) - \frac{m}{t}$.

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

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Value? $f_0(x) + \sum_{i=1} \lambda_i f_i(x) = f_0(x) - \frac{m}{t}$.

Central point $x(t)$ within $\frac{m}{t}$ of optimal!!!!

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

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Fix λ, optimize for x^* give valid lower bound on solution.
Optimality Condition.
Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0$.

Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$! Same optimal point!

Value? $f_0(x) + \sum_{i=1} \lambda_i f_i(x) = f_0(x) - \frac{m}{t}$.

Central point $x(t)$ within $\frac{m}{t}$ of optimal!!!!
$L(\lambda, x(t)) \geq f_0(x) - \frac{m}{t}$

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

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Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$! Same optimal point!

Value? $f_0(x) + \sum_{i=1} \lambda_i f_i(x) = f_0(x) - \frac{m}{t}$.

Central point $x(t)$ within $\frac{m}{t}$ of optimal!!!!
$L(\lambda, x(t)) \geq f_0(x) - \frac{m}{t} \implies \min_x L(\lambda, x) + \frac{m}{t} \geq f_0(x)$

Lagrangian Dual and Central Path.

$\min tf_0(x) - \sum_{i=1} \ln(-f_i(x))$

Optimality condition?

Derivative: $t\nabla f_0(x) + \sum_{i=1} \frac{\nabla f_i(x)}{f_i(x)} = 0$

Or, $\nabla f_0(x) = -\sum_{i=1} \frac{\nabla f_i(x)}{tf_i(x)}$ (Opposing force fields.)

Recall, Lagrangian: $L(\lambda, x) = f_0(x) + \sum_i \lambda_i f_i(x)$.

Fix λ, optimize for x^* give valid lower bound on solution.

Optimality Condition.

Derivative: $\nabla f_0(x) + \sum_{i=1} \lambda_i \nabla f_i(x) = 0$.

Take $\lambda_i^{(t)} = -\frac{1}{tf_i(x)}$. $x(t) = x^*(\lambda^{(t)})$! Same optimal point!

Value? $f_0(x) + \sum_{i=1} \lambda_i f_i(x) = f_0(x) - \frac{m}{t}$.

Central point $x(t)$ within $\frac{m}{t}$ of optimal!!!!

$L(\lambda, x(t)) \geq f_0(x) - \frac{m}{t} \implies \min_x L(\lambda, x) + \frac{m}{t} \geq f_0(x)$

$\implies OPT + \frac{m}{t} \geq f_0(x)$

Central path.

$\min_x f_0(x), f_i(x) \leq 0$.

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Optimal: $x(t)$ is feasible.

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$f_0(x(t)) \geq OPT - \frac{m}{t}$

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Algorithm: take $t \to \infty$.

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Finding $x(t)$?

Central path.

$\min_x f_0(x), f_i(x) \leq 0.$

$\min_x tf_0(x) - \sum_{i>0} \ln(-f_i(x))$

Optimal: $x(t)$ is feasible.

$f_0(x(t)) \geq OPT - \frac{m}{t}$

Algorithm: take $t \to \infty$.

Finding $x(t)$?

Next.