Plant Carrots or Peas?
Profit maximization.

Plant Carrots or Peas?
2$ bushel of carrots.
Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.
Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.
Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.
Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.
Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shady land.

Garden has 40 yards of sunny land and 75 yards of shady land.
Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.

Garden has 40 yards of sunny land and 75 yards of shady land.
Plant Carrots or Peas?
2$ bushel of carrots. 4$ for peas.
Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shady land.

Garden has 40 yards of sunny land and 75 yards of shady land.
To pea or not to pea, that is the question!
To pea or not to pea.

4$ for peas.
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$- to pea!
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

\( x_1 \) - to pea! \( x_2 \) to carrot
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$- to pea! $x_2$ to carrot?
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$- to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$
To pea or not to pea.
  4$ for peas. 2$ bushel of carrots.
  $x_1$ to pea! $x_2$ to carrot?
  Money $4x_1 + 2x_2$ maximize
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$ to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$. 

Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel.

100 units of water.

Peas 2 yards/bushel of sunny land. Have 40 sq. yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can't make negative!

$x_1, x_2 \geq 0$.

A linear program.

$\max 4x_1 + 2x_2$

$2x_1 \leq 40$

$3x_2 \leq 75$

$3x_1 + 2x_2 \leq 100$

$x_1, x_2 \geq 0$. 
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$ - to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$- to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\text{max } 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel.
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

\( x_1 \)- to pea! \( x_2 \) to carrot?

Money \( 4x_1 + 2x_2 \) maximize \( \max 4x_1 + 2x_2 \).

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.
To pea or not to pea.
4$ for peas. 2$ bushel of carrots.
$x_1$ - to pea! $x_2$ to carrot?
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.
$x_1$- to pea! $x_2$ to carrot ?
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$ to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$ - to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$ - to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$ - to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can’t make negative!
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.
$x_1$ to pea! $x_2$ to carrot?
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.
$2x_1 \leq 40$
Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.
$3x_2 \leq 75$
Can’t make negative! $x_1, x_2 \geq 0$. 
To pea or not to pea.

4\$ for peas. 2\$ bushel of carrots.
$x_1$ - to pea! $x_2$ to carrot?
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.
$2x_1 \leq 40$
Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.
$3x_2 \leq 75$
Can’t make negative! $x_1, x_2 \geq 0$.
A linear program.
To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

$x_1$ - to pea! $x_2$ to carrot?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can’t make negative! $x_1, x_2 \geq 0$.

A linear program.

$$
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
\text{subject to } & 2x_1 \leq 40 \\
& 3x_2 \leq 75 \\
& 3x_1 + 2x_2 \leq 100 \\
& x_1, x_2 \geq 0
\end{align*}
$$
max $4x_1 + 2x_2$

$2x_1 \leq 40$

$3x_2 \leq 75$

$3x_1 + 2x_2 \leq 100$

$x_1, x_2 \geq 0$

Optimal point?
\[ \text{max } 4x_1 + 2x_2 \]
\[ 2x_1 \leq 40 \]
\[ 3x_2 \leq 75 \]
\[ 3x_1 + 2x_2 \leq 100 \]
\[ x_1, x_2 \geq 0 \]

Optimal point?
Try every point
\[ \begin{align*}
\text{max } & 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*} \]

Optimal point?

Try every point if we only had time!
max $4x_1 + 2x_2$

$2x_1 \leq 40$

$3x_2 \leq 75$

$3x_1 + 2x_2 \leq 100$

$x_1, x_2 \geq 0$

Optimal point?

Try every point if we only had time!

How many points?
\[
\begin{align*}
\text{max } 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Optimal point?
Try every point if we only had time!
How many points?
Real numbers?
\[ \begin{align*}
\max &\ 4x_1 + 2x_2 \\
2x_1 &\leq 40 \\
3x_2 &\leq 75 \\
3x_1 + 2x_2 &\leq 100 \\
x_1, x_2 &\geq 0
\end{align*} \]

Optimal point?
Try every point if we only had time!
How many points?
Real numbers?
Infinite.
\[
\begin{align*}
\text{max } 4x_1 + 2x_2 \\
2x_1 &\leq 40 \\
3x_2 &\leq 75 \\
3x_1 + 2x_2 &\leq 100 \\
x_1, x_2 &\geq 0
\end{align*}
\]

Optimal point?
Try every point if we only had time!
How many points?
Real numbers?
Infinite. Uncountably infinite!
A linear program.
A linear program.

\[
\text{max } 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\]
A linear program.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Optimal point?
A linear program.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Optimal point?
A linear program.

\[ \max 4x_1 + 2x_2 \]
\[ 2x_1 \leq 40 \]
\[ 3x_2 \leq 75 \]
\[ 3x_1 + 2x_2 \leq 100 \]
\[ x_1, x_2 \geq 0 \]

Optimal point?
A linear program.

\[
\begin{align*}
\text{max} & \quad 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Optimal point?
A linear program.

\[
\begin{align*}
\max & \quad 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Optimal point?
Feasible Region.

Convex.
Any two points in region connected by a line in region.

Algebraically:
If $x$ and $x'$ satisfy constraint: $ax \leq b$ and $ax' \leq b$, $x'' = \alpha x + (1 - \alpha)x' \rightarrow ax'' \leq b$. 
Feasible Region.

Convex.

Any two points in region connected by a line in region.

Algebraically:
If $x$ and $x'$ satisfy constraint:
$$ax \leq b \text{ and } ax' \leq b,$$

then $x'' = \alpha x + (1 - \alpha)x' \rightarrow ax'' \leq b.$
Feasible Region.

Convex.

Any two points in region connected by a line in region.
Feasible Region.

Convex.

Any two points in region connected by a line in region. Algebraically:
Feasible Region.

Convex.

Any two points in region connected by a line in region.

Algebraically:

If $x$ and $x'$ satisfy constraint: $ax \leq b$ and $ax' \leq b$, 

Feasible Region.

Convex.

Any two points in region connected by a line in region.

Algebraically:
If $x$ and $x'$ satisfy constraint: $ax \leq b$ and $ax' \leq b$,

$$x'' = \alpha x + (1 - \alpha)x'$$
Convex.

Any two points in region connected by a line in region.

Algebraically:

If \(x\) and \(x'\) satisfy constraint: \(ax \leq b\) and \(ax' \leq b\),

\[x'' = \alpha x + (1 - \alpha)x' \rightarrow ax'' \leq b.\]
Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

Choose best.

$O\left(\frac{m^2}{2}\right)$ if $m$ constraints and 2 variables.

For $n$ variables, $m$ constraints, how many?

$nm$?

$\left(\frac{m}{n}\right)$?

$n + m$?

$\left(\frac{m}{n}\right)$?

Finite!!!!!!

Exponential in the number of variables.
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex!
Choose best.

$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables, $m$ constraints, how many?
$n m^2$?
$(m n)$?
$n + m$?
Finite!!!!!!
Exponential in the number of variables.
Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

Choose best.

\[ O(m^2) \] if \( m \) constraints and 2 variables.

For \( n \) variables, \( m \) constraints, how many?

\[ nm \?

\[ (m+n) \?

Finite!!!

Exponential in the number of variables.
Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

Choose best.

$O\left(\binom{m^2}{2}\right)$ if $m$ constraints and 2 variables.

For $n$ variables, $m$ constraints, how many?

$\binom{m}{n}$?

$\binom{m^2}{2}$?

$nm$?

$\left(\binom{m}{n}\right)$?

Finite!!!!!!

Exponential in the number of variables.
Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

Choose best.

\[ O(\binom{m}{2}) \] if \( m \) constraints and 2 variables.

For \( n \) variables, \( m \) constraints, how many?

\[ \binom{mn}{n+m} \]?

Finite!!!!!!

Exponential in the number of variables.
Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

Choose best.

$O(m^2)$ if $m$ constraints and 2 variables.

For $n$ variables, $m$ constraints, how many?

$nm$? $(mn)$?

Finite!!!!!!

Exponential in the number of variables.
Optimal at pointy part of feasible region! 

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

Choose best.

$O \left( m^2 \right)$ if $m$ constraints and 2 variables.

For $n$ variables, $m$ constraints, how many?

$nm$?

$\left( mn \right)^2$?

$\left( mn \right)$?

Finite!!!!!!

Exponential in the number of variables.
Optimal at pointy part of feasible region!
Optimal at pointy part of feasible region!
Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

$O(m^2)$ if $m$ constraints and 2 variables.

For $n$ variables, $m$ constraints, how many?

$(m^n)$?? $(m+n)$??

Finite!!!!!!
Exponential in the number of variables.
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!

\[
O(m_2) \quad \text{if} \quad m \quad \text{constraints and 2 variables.}
\]

For \( n \) variables, \( m \) constraints, how many?

\[
O(n^m) \quad \text{or} \quad O(m^n)
\]

Finite!!!!!!
Exponential in the number of variables.
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex!

Choose best.

$O(m^2)$ if $m$ constraints and 2 variables.

For $n$ variables, $m$ constraints, how many?

$nm$?

$\binom{m}{n}$?

$n+m$?

$\binom{m}{n}$

Finite!!!!!!

Exponential in the number of variables.
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables, $m$ constraints, how many?
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables, $m$ constraints, how many? $nm$? $\binom{m}{n}$? $n + m$?
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables, $m$ constraints, how many?
$nm$? $\binom{m}{n}$? $n + m$?
$\binom{m}{n}$
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables, $m$ constraints, how many?
$nm? \binom{m}{n}? n + m$?
$\binom{m}{n}$
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables, $m$ constraints, how many?
$nm$?
$\binom{m}{n}$?
$n + m$?
$\binom{m}{n}$?
Finite!!!!!!
Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if $m$ constraints and 2 variables.
For $n$ variables, $m$ constraints, how many?
$nm$? $\binom{m}{n}$? $n + m$?
$\binom{m}{n}$
Finite!!!!!!
Exponential in the number of variables.
Simplex: Start at vertex.

Maximise \( 4x_1 + 2x_2 \)

- \( 2x_1 \leq 40 \)
- \( 3x_2 \leq 75 \)
- \( 3x_1 + 2x_2 \leq 100 \)
- \( x_1, x_2 \geq 0 \)
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}

Simplex: Start at vertex. Move to better neighboring vertex.

Duality: Add blue equations to get objective function?
\[ \frac{1}{2} \text{ times first plus second.} \]
Get \[ 4x_1 + 2x_2 \leq 120. \]
Every solution must satisfy this inequality!
Objective value: 120.
Can we do better?
No!
Dual problem: add equations to get best upper bound.
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
\text{subject to } & 2x_1 \leq 40 \\
& 3x_2 \leq 75 \\
& 3x_1 + 2x_2 \leq 100 \\
& x_1, x_2 \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.
(0,0) objective 0.

\[
\text{max } 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.

(0,0) objective 0. \(\rightarrow\) (0,25) objective 50.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.
(0,0) objective 0. → (0,25) objective 50. 
→ (16\(\frac{2}{3}\),25) objective 115\(\frac{2}{3}\)

\[
\text{max } 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.
(0,0) objective 0. → (0,25) objective 50.
→ (16\(\frac{2}{3}\),25) objective 115\(\frac{2}{3}\)
→ (20,20) objective 120.

\[
\begin{align*}
\text{max } 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.
(0,0) objective 0. \(\rightarrow\) (0,25) objective 50.
\(\rightarrow\) \((16\frac{2}{3},25)\) objective 115\(\frac{2}{3}\)
\(\rightarrow\) (20,20) objective 120.
Duality:
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.

(0,0) objective 0. → (0,25) objective 50. 
→ (16\(\frac{2}{3}\), 25) objective 115\(\frac{2}{3}\) 
→ (20,20) objective 120.

Duality:
Add blue equations to get objective function?

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
\text{s.t. } & 2x_1 \leq 40 \\
& 3x_2 \leq 75 \\
& 3x_1 + 2x_2 \leq 100 \\
& x_1, x_2 \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.

(0,0) objective 0. → (0,25) objective 50.
→ (16\(\frac{2}{3}\),25) objective 115\(\frac{2}{3}\)
→ (20,20) objective 120.

Duality:
Add blue equations to get objective function?
1/2 times first plus second.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
\text{subject to } & 2x_1 \leq 40 \\
& 3x_2 \leq 75 \\
& 3x_1 + 2x_2 \leq 100 \\
& x_1, x_2 \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.

(0,0) objective 0. → (0,25) objective 50.
→ (16\(\frac{2}{3}\), 25) objective 115\(\frac{2}{3}\)
→ (20,20) objective 120.

Duality:
Add blue equations to get objective function?
1/2 times first plus second.
Get \(4x_1 + 2x_2 \leq 120\).
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.
(0,0) objective 0. \(\rightarrow\) (0,25) objective 50.
\(\rightarrow\) (16\(\frac{2}{3}\),25) objective 115\(\frac{2}{3}\)
\(\rightarrow\) (20,20) objective 120.
Duality:
Add blue equations to get objective function?
1/2 times first plus second.
Get \(4x_1 + 2x_2 \leq 120\). Every solution must satisfy this inequality!

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.

(0,0) objective 0. \(\rightarrow\) (0,25) objective 50.
\(\rightarrow\) (16\(\frac{2}{3}\),25) objective 115\(\frac{2}{3}\)
\(\rightarrow\) (20,20) objective 120.

Duality:
Add blue equations to get objective function?
1/2 times first plus second.
Get \(4x_1 + 2x_2 \leq 120\). Every solution must satisfy this inequality!
Objective value: 120.
Can we do better?

\[
\begin{align*}
\text{max } & \quad 4x_1 + 2x_2 \\
2x_1 & \leq 40 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. This example.

(0,0) objective 0. → (0,25) objective 50.
→ (16\frac{2}{3},25) objective 115\frac{2}{3}
→ (20,20) objective 120.

Duality:
Add blue equations to get objective function?
1/2 times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Objective value: 120.
Can we do better? No!
Dual problem: add equations to get best upper bound.
Duality.

\[
\begin{align*}
\text{max } & \quad x_1 + 8x_2 \\
\text{s.t. } & \quad x_1 \leq 4 \\
& \quad x_2 \leq 3 \\
& \quad x_1 + 2x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Duality.

\[ \text{max } x_1 + 8x_2 \]

\[ x_1 \leq 4 \]
\[ x_2 \leq 3 \]
\[ x_1 + 2x_2 \leq 7 \]
\[ x_1, x_2 \geq 0 \]

One Solution: \( x_1 = 1, x_2 = 3. \)
Duality.

\[ \text{max } x_1 + 8x_2 \]
\[ x_1 \leq 4 \]
\[ x_2 \leq 3 \]
\[ x_1 + 2x_2 \leq 7 \]
\[ x_1, x_2 \geq 0 \]

One Solution: \( x_1 = 1, x_2 = 3 \). Value is 25.
Duality.

\[
\begin{align*}
\text{max } & x_1 + 8x_2 \\
\text{s.t. } & x_1 \leq 4 \\
& x_2 \leq 3 \\
& x_1 + 2x_2 \leq 7 \\
& x_1, x_2 \geq 0 \\
\end{align*}
\]

One Solution: \( x_1 = 1, x_2 = 3 \). Value is 25.

Best possible?
Duality.

\[
\begin{align*}
\max & \quad x_1 + 8x_2 \\
\text{s.t.} & \quad x_1 \leq 4 \\
& \quad x_2 \leq 3 \\
& \quad x_1 + 2x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

One Solution: \( x_1 = 1, x_2 = 3 \). Value is 25.

Best possible?

For any solution.
\( x_1 \leq 4 \) and \( x_2 \leq 3 \) ..
Duality.

\[
\begin{align*}
\text{max } x_1 + 8x_2 \\
& x_1 \leq 4 \\
& x_2 \leq 3 \\
& x_1 + 2x_2 \leq 7 \\
& x_1, x_2 \geq 0
\end{align*}
\]

One Solution: \( x_1 = 1, x_2 = 3 \). Value is 25.

Best possible?

For any solution.
\( x_1 \leq 4 \) and \( x_2 \leq 3 \).

\[
\begin{align*}
& \text{so } x_1 + 8x_2 \leq 4 + 8(3) = 28. \\
& \text{Added equation 1 and 8 times equation 2 yields bound on objective.}
\end{align*}
\]
Duality.

\[
\begin{align*}
\text{max } & x_1 + 8x_2 \\
x_1 & \leq 4 \\
x_2 & \leq 3 \\
x_1 + 2x_2 & \leq 7 \\
x_1, x_2 & \geq 0
\end{align*}
\]

One Solution: \( x_1 = 1, x_2 = 3 \). Value is 25.

Best possible?

For any solution.
\( x_1 \leq 4 \) and \( x_2 \leq 3 \) ..

....so \( x_1 + 8x_2 \leq 4 + 8(3) = 28 \).

Added equation 1 and 8 times equation 2 yields bound on objective..

Better solution?
Duality.

\[
\text{max } x_1 + 8x_2 \\
\text{s.t. } \\
x_1 \leq 4 \\
x_2 \leq 3 \\
x_1 + 2x_2 \leq 7 \\
x_1, x_2 \geq 0
\]

One Solution: \( x_1 = 1, x_2 = 3 \). Value is 25.

Best possible?

For any solution:
\( x_1 \leq 4 \) and \( x_2 \leq 3 \).

....so \( x_1 + 8x_2 \leq 4 + 8(3) = 28 \).

Added equation 1 and 8 times equation 2 yields bound on objective.

Better solution?
Better upper bound?
Duality.

\[
\begin{align*}
\text{max } x_1 + 8x_2 \\
x_1 &\leq 4 \\
x_2 &\leq 3 \\
x_1 + 2x_2 &\leq 7 \\
x_1, x_2 &\geq 0
\end{align*}
\]

One Solution: \( x_1 = 1, x_2 = 3 \). Value is 25.

Best possible?

For any solution.
\( x_1 \leq 4 \) and \( x_2 \leq 3 \).

\[ \text{so } x_1 + 8x_2 \leq 4 + 8(3) = 28. \]

Added equation 1 and 8 times equation 2 yields bound on objective.

Better solution?
Better upper bound?
Duality.

\[
\begin{align*}
\text{max } & \quad x_1 + 8x_2 \\
\text{subject to } & \quad x_1 \leq 4 \\
& \quad x_2 \leq 3 \\
& \quad x_1 + 2x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0 \\
\end{align*}
\]

Solution value: 25.
Duality.

\[
\begin{align*}
\text{max } x_1 + 8x_2 & \\
x_1 & \leq 4 \\
x_2 & \leq 3 \\
x_1 + 2x_2 & \leq 7 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
Duality.

\[
\begin{align*}
\text{max } x_1 + 8x_2 \\
 x_1 \leq 4 \\
 x_2 \leq 3 \\
 x_1 + 2x_2 \leq 7 \\
 x_1, x_2 \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
\[
x_1 + 8x_2 \leq 4 + 24 = 28.
\]
Duality.

\[
\begin{align*}
\text{max } & \quad x_1 + 8x_2 \\
\text{s.t. } & \quad x_1 \leq 4 \\
& \quad x_2 \leq 3 \\
& \quad x_1 + 2x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives.. \[x_1 + 8x_2 \leq 4 + 24 = 28.\]
Better way to add equations to get bound on function?
Duality.

\[
\begin{align*}
\text{max } x_1 + 8x_2 \\
\quad x_1 \leq 4 \\
\quad x_2 \leq 3 \\
\quad x_1 + 2x_2 \leq 7 \\
\quad x_1, x_2 \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
\[x_1 + 8x_2 \leq 4 + 24 = 28.\]

Better way to add equations to get bound on function?
Sure:
Duality.

\[
\begin{align*}
\text{max } & x_1 + 8x_2 \\
\text{s.t. } & x_1 \leq 4 \\
& x_2 \leq 3 \\
& x_1 + 2x_2 \leq 7 \\
& x_1, x_2 \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives.. 
\[x_1 + 8x_2 \leq 4 + 24 = 28.\]

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
Duality.

\[
\begin{align*}
\text{max } x_1 + 8x_2 \\
x_1 &\leq 4 \\
x_2 &\leq 3 \\
x_1 + 2x_2 &\leq 7 \\
x_1, x_2 &\geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
\[x_1 + 8x_2 \leq 4 + 24 = 28.\]

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
\[x_1 + 8x_2 \leq 6(3) + 7 = 25.\]
Duality.

\[
\begin{align*}
\max & \quad x_1 + 8x_2 \\
& \quad x_1 \leq 4 \\
& \quad x_2 \leq 3 \\
& \quad x_1 + 2x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
\[
x_1 + 8x_2 \leq 4 + 24 = 28.
\]

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
\[
x_1 + 8x_2 \leq 6(3) + 7 = 25.
\]

Thus, the value is at most 25.
Duality.

\[
\begin{align*}
\text{max } & \quad x_1 + 8x_2 \\
\text{subject to } & \quad x_1 \leq 4 \\
& \quad x_2 \leq 3 \\
& \quad x_1 + 2x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
\[
x_1 + 8x_2 \leq 4 + 24 = 28.
\]

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
\[
x_1 + 8x_2 \leq 6(3) + 7 = 25.
\]

Thus, the value is at most 25.
The upper bound is same as solution!
Duality.

\[
\begin{align*}
\max & \quad x_1 + 8x_2 \\
\text{s.t.} & \quad x_1 \leq 4 \\
& \quad x_2 \leq 3 \\
& \quad x_1 + 2x_2 \leq 7 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
\[
x_1 + 8x_2 \leq 4 + 24 = 28.
\]
Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
\[
x_1 + 8x_2 \leq 6(3) + 7 = 25.
\]
Thus, the value is at most 25.
The upper bound is same as solution!
Proof of optimality!
Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.
Duality: example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Will this always work?
Idea: Add up positive linear combination of inequalities to "get" upper bound on optimization function.

Will this always work?

How to find best upper bound?
Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 3$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + 2x_2 \leq 7$</td>
</tr>
</tbody>
</table>

Adding equations thusly...
Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 3$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + 2x_2 \leq 7$</td>
</tr>
</tbody>
</table>

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$
Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 3$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + 2x_2 \leq 7$</td>
</tr>
</tbody>
</table>

Adding equations thusly...

$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3$.

The left hand side should “dominate” optimization function:
Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 3$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + 2x_2 \leq 7$</td>
</tr>
</tbody>
</table>

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$  
The left hand side should “dominate” optimization function:

If $y_1, y_2, y_3 \geq 0$
Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 3$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + 2x_2 \leq 7$</td>
</tr>
</tbody>
</table>

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$  

The left hand side should “dominate” optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 3$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + 2x_2 \leq 7$</td>
</tr>
</tbody>
</table>

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$  

The left hand side should “dominate” optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$
Duality: computing upper bound.

Best Upper Bound.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 4$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 3$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + 2x_2 \leq 7$</td>
</tr>
</tbody>
</table>

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$ 

The left hand side should “dominate” optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

Find best $y_i$’s to minimize upper bound?
The dual, the dual, the dual.
Find best $y_i$’s to minimize upper bound?

Again: If you find $y_1$, $y_2$, $y_3 \geq 0$ and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$x_1 + 8x_2 \leq 4$
$y_1 + 3y_2 + 7y_3 \leq \min 4$
$y_1 + y_3 \geq 1$
$y_2 + 2y_3 \geq 8$
$y_1, y_2, y_3 \geq 0$

A linear program.
The Dual linear program.
Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.
Value of both is 25! Primal is optimal... and dual is optimal!
The dual, the dual, the dual.

Find best $y_i$’s to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
The dual, the dual, the dual.

Find best $y_i$’s to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
The dual, the dual, the dual.

Find best $y_i$’s to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

A linear program.
The Dual linear program.
Primal: $(x_1, x_2) = (1, 3)$
Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!
Primal is optimal... and dual is optimal!
The dual, the dual, the dual.

Find best \(y_i\)'s to minimize upper bound?

Again: If you find \(y_1, y_2, y_3 \geq 0\) and \(y_1 + y_3 \geq 1\) and \(y_2 + 2y_3 \geq 8\) then.. 
\[
x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3
\]

\[
\min 4y_1 + 3y_2 + 7y_3 \\
y_1 + y_3 \geq 1 \\
y_2 + 2y_3 \geq 8 \\
y_1, y_2, y_3 \geq 0
\]
The dual, the dual, the dual.

Find best $y_i$’s to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

$$\min 4y_1 + 3y_2 + 7y_3$$

$$y_1 + y_3 \geq 1$$

$$y_2 + 2y_3 \geq 8$$

$$y_1, y_2, y_3 \geq 0$$

A linear program.
The dual, the dual, the dual.
Find best $y_i$’s to minimize upper bound?
Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

\[
\min 4y_1 + 3y_2 + 7y_3
\]
\[
y_1 + y_3 \geq 1
\]
\[
y_2 + 2y_3 \geq 8
\]
\[
y_1, y_2, y_3 \geq 0
\]

A linear program.
The Dual linear program.
The dual, the dual, the dual.
Find best \( y_i \)'s to minimize upper bound?

Again: If you find \( y_1, y_2, y_3 \geq 0 \)
and \( y_1 + y_3 \geq 1 \) and \( y_2 + 2y_3 \geq 8 \) then..
\[
x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3
\]

\[
\min 4y_1 + 3y_2 + 7y_3
\]
\[
y_1 + y_3 \geq 1
\]
\[
y_2 + 2y_3 \geq 8
\]
\[
y_1, y_2, y_3 \geq 0
\]

A linear program.
The Dual linear program.

Primal: \((x_1, x_2) = (1, 3)\); Dual: \((y_1, y_2, y_3) = (0, 6, 1)\).
The dual, the dual, the dual.
Find best $y_i$’s to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

$$\min 4y_1 + 3y_2 + 7y_3$$
$$y_1 + y_3 \geq 1$$
$$y_2 + 2y_3 \geq 8$$
$$y_1, y_2, y_3 \geq 0$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!
The dual, the dual, the dual.
Find best $y_i$'s to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$ and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then...

$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

$$\min 4y_1 + 3y_2 + 7y_3$$

$y_1 + y_3 \geq 1$

$y_2 + 2y_3 \geq 8$

$y_1, y_2, y_3 \geq 0$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal
The dual, the dual, the dual.

Find best \( y_i \)'s to minimize upper bound?

Again: If you find \( y_1, y_2, y_3 \geq 0 \) and \( y_1 + y_3 \geq 1 \) and \( y_2 + 2y_3 \geq 8 \) then..
\[ x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3 \]

\[
\begin{align*}
\min 4y_1 + 3y_2 + 7y_3 \\
y_1 + y_3 \geq 1 \\
y_2 + 2y_3 \geq 8 \\
y_1, y_2, y_3 \geq 0
\end{align*}
\]

A linear program.
The Dual linear program.

Primal: \((x_1, x_2) = (1, 3)\); Dual: \((y_1, y_2, y_3) = (0, 6, 1)\).

Value of both is 25!

Primal is optimal ... and dual is optimal!
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max c \cdot x$</td>
<td>$\min y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max } c \cdot x$</td>
<td>$\text{min } y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual $(D)$
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual $(D)$
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal ($P$) $\leq$ dual ($D$)

Feasible ($x, y$)
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual $(D)$

Feasible $(x,y)$

$P(x)$
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{max } c \cdot x )</td>
<td>( \text{min } y^T b )</td>
</tr>
<tr>
<td>( Ax \leq b )</td>
<td>( y^T A \geq c )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td>( y \geq 0 )</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal \( (P) \leq \) dual \( (D) \)

Feasible \( (x, y) \)

\[ P(x) = c \cdot x \]
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max } c \cdot x$</td>
<td>$\text{min } y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal ($P$) $\leq$ dual ($D$)

Feasible $(x, y)$

$P(x) = c \cdot x \leq y^T Ax$
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual $(D)$

Feasible $(x, y)$

$$P(x) = c \cdot x \leq y^T A x \leq y^T b \cdot x = D(y).$$
The dual.

In general.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

**Theorem:** If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal ($P$) $\leq$ dual ($D$)

Feasible ($x, y$)

$P(x) = c \cdot x \leq y^T Ax \leq y^T b \cdot x = D(y)$.

Strong Duality: next lecture, previous lectures maybe?
### Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{max } c \cdot x )</td>
<td>( \text{min } y^T b )</td>
</tr>
<tr>
<td>( Ax \leq b )</td>
<td>( y^T A \geq c )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td>( y \geq 0 )</td>
</tr>
</tbody>
</table>

Given \( A, b, c \), and feasible solutions \( x \) and \( y \).
Complementary Slackness

\begin{align*}
\text{Primal LP} & \quad \text{Dual LP} \\
\max c \cdot x & \quad \min y^T b \\
Ax \leq b & \quad y^T A \geq c \\
x \geq 0 & \quad y \geq 0
\end{align*}

Given $A, b, c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if
\[ x_i(c_i - (y^T A)_i) = 0, \quad \text{and} \quad y_j(b_j - (Ax)_j). \]
Complementary Slackness

Primal LP | Dual LP
---|---
max $c \cdot x$ | min $y^T b$
$Ax \leq b$ | $y^T A \geq c$
$x \geq 0$ | $y \geq 0$

Given $A, b, c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if

\[ x_i(c_i - (y^T A)_i) = 0, \quad \text{and} \quad y_j(b_j - (Ax)_j). \]

\[ x_i(c_i - (y^T A)_i) = 0 \rightarrow \]
## Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Given $A, b, c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if

$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

\[
x_i(c_i - (y^T A)_i) = 0 \rightarrow \\
\sum_i(c_i - (y^T A)_i)x_i
\]
Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Given $A, b, c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if

$$x_i(c_i - (y^T A)_i) = 0,$$
$$y_j(b_j - (Ax)_j).$$

$$x_i(c_i - (y^T A)_i) = 0 \rightarrow \sum_i(c_i - (y^T A)_i)x_i = cx - y^T Ax$$
Complementary Slackness

Primal LP

\[
\begin{align*}
\text{max } c \cdot x \\
Ax & \leq b \\
x & \geq 0
\end{align*}
\]

Dual LP

\[
\begin{align*}
\text{min } y^T b \\
y^T A & \geq c \\
y & \geq 0
\end{align*}
\]

Given \( A, b, c \), and feasible solutions \( x \) and \( y \).

Solutions \( x \) and \( y \) are both optimal if and only if

\[
x_i(c_i - (y^T A)_i) = 0, \text{ and } y_j(b_j - (Ax)_j).
\]

\[
x_i(c_i - (y^T A)_i) = 0 \rightarrow \\
\sum_i(c_i - (y^T A)_i)x_i = cx - y^T Ax \rightarrow cx = y^T Ax.
\]
Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Given $A$, $b$, $c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if

$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

\[
x_i(c_i - (y^T A)_i) = 0 \rightarrow \\
\sum_i (c_i - (y^T A)_i)x_i = cx - y^T Ax \rightarrow cx = y^T Ax.
\]

$y_j(b_j - (Ax)_j) = 0 \rightarrow$
Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Given $A, b, c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if

$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

$x_i(c_i - (y^T A)_i) = 0 \rightarrow
\sum_i (c_i - (y^T A)_i)x_i = cx - y^T Ax \rightarrow cx = y^T Ax.$

$y_j(b_j - (Ax)_j) = 0 \rightarrow
\sum_i y_j(b_j - (Ax)_j)$
Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max c \cdot x$</td>
<td>$\min y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Given $A$, $b$, $c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if

$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

$x_i(c_i - (y^T A)_i) = 0 \rightarrow$

$\sum_i (c_i - (y^T A)_i)x_i = cx - y^T Ax \rightarrow cx = y^T Ax.$

$y_j(b_j - (Ax)_j) = 0 \rightarrow$

$\sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax$
Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max c \cdot x )</td>
<td>( \min y^T b )</td>
</tr>
<tr>
<td>( Ax \leq b )</td>
<td>( y^T A \geq c )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td>( y \geq 0 )</td>
</tr>
</tbody>
</table>

Given \( A, b, c \), and feasible solutions \( x \) and \( y \).

Solutions \( x \) and \( y \) are both optimal if and only if
\[
x_i(c_i - (y^T A)_i) = 0, \text{ and } y_j(b_j - (Ax)_j).
\]

\[
x_i(c_i - (y^T A)_i) = 0 \rightarrow \\
\sum_i(c_i - (y^T A)_i)x_i = cx - y^T Ax \rightarrow cx = y^T Ax.
\]

\[
y_j(b_j - (Ax)_j) = 0 \rightarrow \\
\sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \rightarrow by = y^T Ax.
\]
Complementary Slackness

**Primal LP**  
\[ \max c \cdot x \]  
\[ Ax \leq b \]  
\[ x \geq 0 \]

**Dual LP**  
\[ \min y^T b \]  
\[ y^T A \geq c \]  
\[ y \geq 0 \]

Given \( A, b, c \), and feasible solutions \( x \) and \( y \).

Solutions \( x \) and \( y \) are both optimal if and only if  
\[ x_i(c_i - (y^T A)_i) = 0, \text{ and } y_j(b_j - (Ax)_j). \]

\[ x_i(c_i - (y^T A)_i) = 0 \rightarrow \sum_i (c_i - (y^T A)_i)x_i = cx - y^T Ax \rightarrow cx = y^T Ax. \]

\[ y_j(b_j - (Ax)_j) = 0 \rightarrow \sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \rightarrow by = y^T Ax. \]

\[ cx = by. \]
Complementary Slackness

Primal LP
max \( c \cdot x \)
\[ Ax \leq b \]
x \( \geq 0 \)

Dual LP
min \( y^T b \)
\[ y^T A \geq c \]
y \( \geq 0 \)

Given \( A, b, c \), and feasible solutions \( x \) and \( y \).

Solutions \( x \) and \( y \) are both optimal if and only if
\[ x_i(c_i - (y^T A)_i) = 0, \text{ and } y_j(b_j - (Ax)_j). \]

\[ x_i(c_i - (y^T A)_i) = 0 \Rightarrow \]
\[ \sum_i(c_i - (y^T A)_i)x_i = cx - y^T Ax \Rightarrow cx = y^T Ax. \]

\[ y_j(b_j - (Ax)_j) = 0 \Rightarrow \]
\[ \sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \Rightarrow by = y^T Ax. \]
\[ cx = by. \]

If both are feasible, \( cx \leq by \), so must be optimal.
Complementary Slackness

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Given $A, b, c$, and feasible solutions $x$ and $y$.

Solutions $x$ and $y$ are both optimal if and only if $x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

$$x_i(c_i - (y^T A)_i) = 0 \rightarrow \sum_i(x_i(c_i - (y^T A)_i))x_i = cx - y^T Ax \rightarrow cx = y^T Ax.$$  

$$y_j(b_j - (Ax)_j) = 0 \rightarrow \sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \rightarrow by = y^T Ax.$$  

$cx = by$.

If both are feasible, $cx \leq by$, so must be optimal.

In words: nonzero dual variables only for tight constraints!
Again: simplex

Simplex: Start at vertex.

\[
\begin{align*}
\text{max } 4x_1 + 2x_2 \\
3x_1 &\leq 60 \\
3x_2 &\leq 75 \\
3x_1 + 2x_2 &\leq 100 \\
x_1, x_2 &\geq 0
\end{align*}
\]
Again: simplex

\[
\text{max } 4x_1 + 2x_2 \\
3x_1 \leq 60 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\]

Simplex: Start at vertex. Move to better neighboring vertex.
Again: simplex

\[
\begin{align*}
\text{max}\ 4x_1 + 2x_2 \\
3x_1 &\leq 60 \\
3x_2 &\leq 75 \\
3x_1 + 2x_2 &\leq 100 \\
x_1, x_2 &\geq 0
\end{align*}
\]

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor.
Again: simplex

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:

\[
\begin{align*}
\text{max } & \quad 4x_1 + 2x_2 \\
& \quad 3x_1 \leq 60 \\
& \quad 3x_2 \leq 75 \\
& \quad 3x_1 + 2x_2 \leq 100 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Again: simplex

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function?

\[
\begin{align*}
\text{max } 4x_1 + 2x_2 \\
3x_1 &\leq 60 \\
3x_2 &\leq 75 \\
3x_1 + 2x_2 &\leq 100 \\
x_1, x_2 &\geq 0
\end{align*}
\]
Again: simplex

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function? 1/3 times first plus second.

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function? 
1/3 times first plus second. Get $4x_1 + 2x_2 \leq 120$. 

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Again: simplex

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function?
1/3 times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!

$$\begin{align*}
\text{max } 4x_1 + 2x_2 \\
3x_1 &\leq 60 \\
3x_2 &\leq 75 \\
3x_1 + 2x_2 &\leq 100 \\
x_1, x_2 &\geq 0
\end{align*}$$
Again: simplex

\[
\begin{align*}
\text{max } 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality: Add blue equations to get objective function? 1/3 times first plus second. Get \(4x_1 + 2x_2 \leq 120\). Every solution must satisfy this inequality!
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function? 1/3 times first plus second. Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Geometrically and Complementary slackness:
Again: simplex

\[
\begin{align*}
\text{max } & 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function? 1/3 times first plus second. Get \(4x_1 + 2x_2 \leq 120\). Every solution must satisfy this inequality!

Geometrically and Complementary slackness:
Add tight constraints to “dominate objective function.”
Again: simplex

\[
\begin{align*}
\text{max} & \quad 4x_1 + 2x_2 \\
3x_1 & \leq 60 \\
3x_2 & \leq 75 \\
3x_1 + 2x_2 & \leq 100 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function? 1/3 times first plus second. Get \(4x_1 + 2x_2 \leq 120\). Every solution must satisfy this inequality!

Geometrically and Complementary slackness:
Add tight constraints to “dominate objective function.”

Don’t add this equation!
Again: simplex

Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:
Add blue equations to get objective function? 1/3 times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!

Geometrically and Complementary slackness:
Add tight constraints to “dominate objective function.”

Don’t add this equation! Shifts.

$\max 4x_1 + 2x_2$

$3x_1 \leq 60$

$3x_2 \leq 75$

$3x_1 + 2x_2 \leq 100$

$x_1, x_2 \geq 0$
Example: review.

\[
\begin{align*}
\text{max } x_1 + 8x_2 & \quad \text{min } 4y_1 + 3y_2 + 7y_3 \\
 x_1 & \leq 4 & y_1 + y_3 & \geq 1 \\
 x_2 & \leq 3 & y_2 + 2y_3 & \geq 8 \\
 x_1 + 2x_2 & \leq 7 & x_1, x_2 & \geq 0 \\
 y_1, y_2, y_3 & \geq 0
\end{align*}
\]

“Matrix form”

\[
\begin{align*}
\text{max}[1,8] \cdot [x_1, x_2] & \quad \text{min}[4,3,7] \cdot [y_1, y_2, y_3] \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} [x_1] & \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} [y_1, y_2, y_3] & \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix} \\
[x_1, x_2] & \geq 0 \\
[y_1, y_2, y_3] & \geq 0
\end{align*}
\]
Matrix equations.

\[
\begin{align*}
\max [1, 8] \cdot [x_1, x_2] & \quad \min [4, 3, 7] \cdot [y_1, y_2, y_3] \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \leq \\
\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} & \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
[x_1, x_2] & \geq 0 & [y_1, y_2, y_3] & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} & \geq \\
& & & \begin{bmatrix} 1 \\ 8 \end{bmatrix} & \quad [y_1, y_2, y_3] \geq 0
\end{align*}
\]

We can rewrite the above in matrix form.

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \quad c = [1, 8] \quad b = [4, 3, 7]
\]

The primal is \( Ax \leq b, \max c \cdot x, x \geq 0. \)

The dual is \( y^T A \geq c, \min b \cdot y, y \geq 0. \)
Rules for School...

or...”Rules for taking duals”

Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>
Rules for School...

or...”Rules for taking duals”

Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{max } c \cdot x )</td>
<td>( \text{min } y^T b )</td>
</tr>
<tr>
<td>( Ax \leq b )</td>
<td>( y^T A \geq c )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td>( y \geq 0 )</td>
</tr>
</tbody>
</table>

Standard:
Rules for School...

or...”Rules for taking duals”

Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Standard:
Rules for School...

or...”Rules for taking duals”
Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Standard:

$Ax \leq b, \max cx, x \geq 0 \leftrightarrow y^T A \geq c, \min by, y \geq 0.$
Rules for School...
or...”Rules for taking duals”
Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max } c \cdot x$</td>
<td>$\text{min } y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Standard:

$Ax \leq b, \text{max } cx, x \geq 0 \iff y^T A \geq c, \text{min } by, y \geq 0.$

$\text{min } \iff \text{max}$
Rules for School...

or..."Rules for taking duals"
Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Standard:

$Ax \leq b, \max cx, x \geq 0 \iff y^T A \geq c, \min by, y \geq 0.$

min $\iff$ max

$\geq \iff \leq$
Rules for School...
or...”Rules for taking duals”
Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max c \cdot x$</td>
<td>$\min y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Standard:

$Ax \leq b, \max cx, x \geq 0 \iff y^T A \geq c, \min by, y \geq 0.$

$\min \leftrightarrow \max$

$\geq \leftrightarrow \leq$

“inequalities” $\leftrightarrow$ “nonnegative variables”
Rules for School...
or...”Rules for taking duals”
Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Standard:

$Ax \leq b, \max cx, x \geq 0 \leftrightarrow y^T A \geq c, \min by, y \geq 0$.

min $\leftrightarrow$ max

$\geq \leftrightarrow \leq$

“inequalities” $\leftrightarrow$ “nonnegative variables”

“nonnegative variables” $\leftrightarrow$ “inequalities”
**Rules for School...**

or...”Rules for taking duals”

Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max ( c \cdot x )</td>
<td>min ( y^T b )</td>
</tr>
<tr>
<td>( Ax \leq b )</td>
<td>( y^T A \geq c )</td>
</tr>
<tr>
<td>( x \geq 0 )</td>
<td>( y \geq 0 )</td>
</tr>
</tbody>
</table>

Standard:

\[ Ax \leq b, \max cx, x \geq 0 \iff y^T A \geq c, \min by, y \geq 0. \]

\( \min \leftrightarrow \max \)

\( \geq \leftrightarrow \leq \)

“inequalities” \(\leftrightarrow\) “nonnegative variables”

“nonnegative variables” \(\leftrightarrow\) “inequalities”

Another useful trick: Equality constraints.
Rules for School...
or…”Rules for taking duals”
Canonical Form.

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $c \cdot x$</td>
<td>min $y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>

Standard:

$Ax \leq b, \max cx, x \geq 0 \iff y^T A \geq c, \min by, y \geq 0.$

min $\leftrightarrow$ max

$\geq \leftrightarrow \leq$

“inequalities” $\leftrightarrow$ “nonnegative variables”

“nonnegative variables” $\leftrightarrow$ “inequalities”

Another useful trick: Equality constraints. “equalities” $\leftrightarrow$

“unrestricted variables.”
Maximum Weight Matching.

Bipartite Graph $G = (V, E), \ w : E \to Z.$
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \to Z$.
Find maximum weight perfect matching.
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \to Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_e w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1$$

$$x_e \geq 0$$

Dual.
Variable for each constraint.
$p_v$ unrestricted.
Constraint for each variable.
$e, p_u + p_v \geq w_e$
Objective function from right hand side.
$\min \sum_v p_v$
$\min \sum v p_v$
$\forall e = (u, v) : p_u + p_v \geq w_e$

Weak duality?
Price function upper bounds matching.

$\sum_{e \in M} w_e x_e \leq \sum_{e=(u,v) \in M} p_u + p_v \leq \sum_v p_v$.

Strong Duality?
Same value solutions.
Hungarian algorithm !!!
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e} w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1$$

$$x_e \geq 0$$

Dual.
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_e w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1$$

$$x_e \geq 0$$

Dual.
Variable for each constraint.
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \to \mathbb{Z}$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e \in E} w_e x_e$$

$$\forall v : \sum_{e = (u, v)} x_e = 1 \quad p_v$$

$$x_e \geq 0$$

Dual.
Variable for each constraint. $p_v$
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow \mathbb{Z}$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_e w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1 \quad p_v$$

$$x_e \geq 0$$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable.
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e} w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1 \quad p_v$$

$$x_e \geq 0$$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side.
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_e w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1 \quad p_v$$

$x_e \geq 0$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e} w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1 \quad \text{ } p_v$$

$$x_e \geq 0$$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$

$$\min \sum_v p_v$$

$$\forall e = (u, v) : \quad p_u + p_v \geq w_e$$
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \to \mathbb{Z}$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e} w_{e} x_{e}$$

$$\forall v : \sum_{e=(u,v)} x_{e} = 1 \quad \text{ } p_{v}$$

$$x_{e} \geq 0$$

Dual.
Variable for each constraint. $p_{v}$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_{v} p_{v}$

$$\min \sum_{v} p_{v}$$

$$\forall e = (u, v) : \quad p_u + p_v \geq w_e$$

Weak duality?
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e} w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1$$  \hspace{1cm} p_v

$$x_e \geq 0$$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum v p_v$

$$\min \sum v p_v$$

$$\forall e = (u, v) : \ p_u + p_v \geq w_e$$

Weak duality? Price function upper bounds matching.
Maximum Weight Matching.

Bipartite Graph \( G = (V, E), \ w: E \to \mathbb{Z} \).
Find maximum weight perfect matching.
Solution: \( x_e \) indicates whether edge \( e \) is in matching.

\[
\max \sum_{e} w_e x_e
\]

\[
\forall v : \sum_{e=(u,v)} x_e = 1
\]

\[p_v\]

\[x_e \geq 0\]

Dual.
Variable for each constraint. \( p_v \) unrestricted.
Constraint for each variable. Edge \( e \), \( p_u + p_v \geq w_e \)
Objective function from right hand side. \( \min \sum_v p_v \)

\[
\min \sum_{v} p_v
\]

\[
\forall e = (u, v) : \ p_u + p_v \geq w_e
\]

Weak duality? Price function upper bounds matching.
\[
\sum_{e \in M} w_e x_e \leq \sum_{e=(u,v) \in M} p_u + p_v \leq \sum_v p_u.
\]
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e} w_e x_e$$
$$\forall v : \sum_{e=(u,v)} x_e = 1$$
$$x_e \geq 0$$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$

$$\min \sum_v p_v$$
$$\forall e = (u, v) : p_u + p_v \geq w_e$$

Weak duality? Price function upper bounds matching.
$\sum_{e \in M} w_e x_e \leq \sum_{e=(u,v) \in M} p_u + p_v \leq \sum_v p_u$.

Strong Duality?
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow \mathbb{Z}$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_{e} w_e x_e$$

$\forall v : \sum_{e=(u,v)} x_e = 1$

$p_v$

$x_e \geq 0$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$

$$\min \sum_v p_v$$

$\forall e = (u, v) : p_u + p_v \geq w_e$

Weak duality? Price function upper bounds matching.

$$\sum_{e \in M} w_e x_e \leq \sum_{e=(u,v) \in M} p_u + p_v \leq \sum_v p_u.$$ 

Strong Duality? Same value solutions.
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_e w_e x_e$$

$\forall v : \sum_{e=(u,v)} x_e = 1 \quad p_v$

$x_e \geq 0$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$

$$\min \sum_v p_v$$

$\forall e = (u, v) : \quad p_u + p_v \geq w_e$

Weak duality? Price function upper bounds matching.
$\sum_{e \in M} w_e x_e \leq \sum_{e=(u,v) \in M} p_u + p_v \leq \sum_v p_u$.

Strong Duality? Same value solutions. Hungarian algorithm
Maximum Weight Matching.

Bipartite Graph $G = (V, E)$, $w : E \rightarrow Z$.
Find maximum weight perfect matching.
Solution: $x_e$ indicates whether edge $e$ is in matching.

$$\max \sum_e w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1$$

$$x_e \geq 0$$

Dual.
Variable for each constraint. $p_v$ unrestricted.
Constraint for each variable. Edge $e$, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$

$$\min \sum_v p_v$$

$$\forall e = (u, v) : p_u + p_v \geq w_e$$

Weak duality? Price function upper bounds matching.
$$\sum_{e \in M} w_e x_e \leq \sum_{e=(u,v) \in M} p_u + p_v \leq \sum_v p_u.$$ 

Strong Duality? Same value solutions. Hungarian algorithm !!!
$x_e$ variable for $e = (u, v)$.

<table>
<thead>
<tr>
<th></th>
<th>$x_e$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_u$</td>
<td>. . 0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>. . 1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>. . 0</td>
<td>1</td>
</tr>
<tr>
<td>$p_v$</td>
<td>. . 1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>. . 0</td>
<td>1</td>
</tr>
<tr>
<td>obj</td>
<td>. . $w_e$</td>
<td>.</td>
</tr>
</tbody>
</table>

Row equation: $\sum x_e = (u, v)$, $x_e = 1$.

Row (dual) variable: $p_u$.

Column variable: $x_e$.

Column (dual) constraint: $p_u + p_v \geq 1$. 

Exercise: objectives?
Matrix View.

$x_e$ variable for $e = (u, v)$.

<table>
<thead>
<tr>
<th></th>
<th>$x_e$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_u$</td>
<td>. . . 0 . .</td>
<td>1</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>. . . 1 . .</td>
<td>1</td>
</tr>
<tr>
<td>obj</td>
<td>. . $w_e$ .</td>
<td></td>
</tr>
</tbody>
</table>

Row equation: $\sum_{e=(u,v)} x_e = 1$. 
**Matrix View.**

$x_e$ variable for $e = (u, v)$.

<table>
<thead>
<tr>
<th></th>
<th>$x_e$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_u$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>⋯</td>
</tr>
<tr>
<td></td>
<td>⋮</td>
<td>⋮</td>
</tr>
<tr>
<td></td>
<td>⋮</td>
<td>⋮</td>
</tr>
<tr>
<td>$p_v$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>⋯</td>
</tr>
<tr>
<td></td>
<td>⋮</td>
<td>⋮</td>
</tr>
<tr>
<td></td>
<td>⋮</td>
<td>⋮</td>
</tr>
<tr>
<td>obj</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>⋮</td>
<td>w_e</td>
</tr>
</tbody>
</table>

Row equation: $\sum_{e=(u,v)} x_e = 1$. Row (dual) variable:
Matrix View.

$x_e$ variable for $e = (u, v)$.

<table>
<thead>
<tr>
<th></th>
<th>$x_e$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_u$</td>
<td>. . . 0 .</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>. . . 1 .</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>. . . 0 .</td>
<td>1</td>
</tr>
<tr>
<td>$p_v$</td>
<td>. . . 0 .</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>. . . 1 .</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>. . . 0 .</td>
<td>1</td>
</tr>
<tr>
<td>obj</td>
<td>. . $w_e$ .</td>
<td>1</td>
</tr>
</tbody>
</table>

Row equation: $\sum_{e=(u,v)} x_e = 1$. Row (dual) variable: $p_u$. 
Matrix View.

$x_e$ variable for $e = (u, v)$.

<table>
<thead>
<tr>
<th></th>
<th>$x_e$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_u$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$p_v$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>obj</td>
<td>$w_e$</td>
<td></td>
</tr>
</tbody>
</table>

Row equation: $\sum_{e=(u,v)} x_e = 1$. Row (dual) variable: $p_u$.

Column variable: $x_e$. 
Matrix View.

$x_e$ variable for $e = (u, v)$.

<table>
<thead>
<tr>
<th></th>
<th>$x_e$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_u$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td></td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>0</td>
</tr>
<tr>
<td>$p_v$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>0</td>
</tr>
<tr>
<td>obj</td>
<td>.</td>
<td>$w_e$</td>
</tr>
</tbody>
</table>

Row equation: $\sum_{e=(u,v)} x_e = 1$. Row (dual) variable: $p_u$.

Column variable: $x_e$. Column (dual) constraint:
Matrix View.

\( x_e \) variable for \( e = (u, v) \).

<table>
<thead>
<tr>
<th></th>
<th>( x_e )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_u )</td>
<td>⋮ ⋮ 0 ⋮ ⋮ 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⋮ ⋮ 1 ⋮ ⋮ 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⋮ ⋮ 0 ⋮ ⋮ 1</td>
<td></td>
</tr>
<tr>
<td>( \rho_v )</td>
<td>⋮ ⋮ 0 ⋮ ⋮ 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⋮ ⋮ 1 ⋮ ⋮ 1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>⋮ ⋮ 0 ⋮ ⋮ 1</td>
<td></td>
</tr>
<tr>
<td>obj</td>
<td>⋮ ⋮ ( w_e ) ⋮ 1</td>
<td></td>
</tr>
</tbody>
</table>

Row equation: \( \sum_{e=(u,v)} x_e = 1 \). Row (dual) variable: \( \rho_u \).
Column variable: \( x_e \). Column (dual) constraint: \( \rho_u + \rho_v \geq 1 \). 

Matrix View.

$x_e$ variable for $e = (u, v)$.

<table>
<thead>
<tr>
<th></th>
<th>$x_e$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$p_u$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$p_v$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>obj</td>
<td>$w_e$</td>
<td>1</td>
</tr>
</tbody>
</table>

Row equation: $\sum_{e=(u,v)} x_e = 1$. Row (dual) variable: $p_u$.
Column variable: $x_e$. Column (dual) constraint: $p_u + p_v \geq 1$.
Exercise: objectives?
Complementary Slackness.

\[
\begin{align*}
\text{max} \sum_{e} w_e x_e \\
\forall v: \sum_{e=(u,v)} x_e &= 1 \\
x_e &\geq 0
\end{align*}
\]

\[
\begin{align*}
p_v \\
x_e &\geq 0
\end{align*}
\]

Dual:

\[
\begin{align*}
\text{min} \sum_{v} p_v \\
\forall e = (u,v): p_u + p_v &\geq w_e
\end{align*}
\]
Complementary Slackness.

\[
\max \sum_{e} w_e x_e\\
\forall v : \sum_{e=(u,v)} x_e = 1\\
\quad x_e \geq 0
\]

Dual:

\[
\min \sum_{v} p_v\\
\forall e = (u, v) : p_u + p_v \geq w_e
\]

Complementary slackness:
Complementary Slackness.

\[
\max \sum_{e} w_e x_e \\
\forall v : \sum_{e=(u,v)} x_e = 1 \quad p_v \\
x_e \geq 0
\]

Dual:

\[
\min \sum_{v} p_v \\
\forall e = (u, v) : \quad p_u + p_v \geq w_e
\]

Complementary slackness:
Only match on tight edges.
Nonzero \( p_u \) on matched \( u \).
Complementary Slackness.

\[
\max \sum_{e} w_e x_e \\
\forall v: \sum_{e=(u,v)} x_e = 1 \quad p_v \\
x_e \geq 0
\]

Dual:

\[
\min \sum_{v} p_v \\
\forall e = (u,v): \quad p_u + p_v \geq w_e
\]

Complementary slackness:
Only match on tight edges.
Nonzero \( p_u \) on matched \( u \).

Constraint for \( u \):
Complementary Slackness.

$$\max \sum_{e} w_e x_e$$

$$\forall v : \sum_{e=(u,v)} x_e = 1 \quad p_v$$

$$x_e \geq 0$$

Dual:

$$\min \sum_{v} p_v$$

$$\forall e = (u, v) : \quad p_u + p_v \geq w_e$$

Complementary slackness:

Only match on tight edges.

Nonzero $p_u$ on matched $u$.

Constraint for $u$:

$$\sum_{e=(u,v)} x_e \leq 1.$$
Complementary Slackness.

\[ \max \sum_{e} w_e x_e \]
\[ \forall v : \sum_{e=(u,v)} x_e = 1 \]
\[ x_e \geq 0 \]

Dual:
\[ \min \sum_{v} p_v \]
\[ \forall e = (u, v) : p_u + p_v \geq w_e \]

Complementary slackness:
Only match on tight edges.
Nonzero \( p_u \) on matched \( u \).

Constraint for \( u \):
\[ \sum_{e=(u,v)} x_e \leq 1 \]
Nonzero \( p_u \) on matched \( u \).
Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. 

Multicommodity Flow.
Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow Z$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$.
Route $D_i$ flow for each $s_i, t_i$ pair,
Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow Z$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. Route $D_i$ flow for each $s_i, t_i$ pair, so every edge has $\leq \mu c(e)$ flow.
Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs
$(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$.
Route $D_i$ flow for each $s_i, t_i$ pair,
so every edge has $\leq \mu c(e)$ flow
with minimum $\mu$. 

variables: $f_p$ flow on path $p$.
$P_i$-set of paths with endpoints $s_i, t_i$.
$\min \mu \forall e : \sum_{p \ni e} f_p \leq \mu c(e) \forall i : \sum_{p \in P_i} f_p = D_i f_p \geq 0$
Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \to \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. Route $D_i$ flow for each $s_i, t_i$ pair, so every edge has $\leq \mu c(e)$ flow with minimum $\mu$. 
Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \rightarrow \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. Route $D_i$ flow for each $s_i, t_i$ pair, so every edge has $\leq \mu c(e)$ flow with minimum $\mu$.

variables: $f_p$ flow on path $p$. $P_i$ -set of paths with endpoints $s_i, t_i$. 
Multicommodity Flow.

Given $G = (V, E)$, and capacity function $c : E \to \mathbb{Z}$, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with demands $D_1, \ldots, D_k$. Route $D_i$ flow for each $s_i, t_i$ pair, so every edge has $\leq \mu c(e)$ flow with minimum $\mu$.

variables: $f_p$ flow on path $p$.
$P_i$ - set of paths with endpoints $s_i, t_i$.

$$\begin{align*}
\min \mu \\
\forall e : \sum_{p \ni e} f_p &\leq \mu c_e \\
\forall i : \sum_{p \in P_i} f_p &= D_i \\
f_p &\geq 0
\end{align*}$$
Take the dual.

\[
\begin{align*}
\min \mu \\
\forall e: \sum_{p \ni e} f_p &\leq \mu c_e \\
\forall i: \sum_{p \in P_i} f_p &= D_i \\
f_p &\geq 0
\end{align*}
\]

Modify to make it \(\geq\), which “go with min.”
Take the dual.

\[
\begin{align*}
\min \mu \\
\forall e : \sum_{p \ni e} f_p & \leq \mu c_e \\
\forall i : \sum_{p \in P_i} f_p & = D_i \\
\quad f_p & \geq 0
\end{align*}
\]

Modify to make it \( \geq \), which “go with min. And only constants on right hand side.
Take the dual.

\[
\min \mu \\
\forall e: \sum_{p \ni e} f_p \leq \mu c_e \\
\forall i: \sum_{p \in P_i} f_p = D_i \\
f_p \geq 0
\]

Modify to make it \( \geq \), which “go with min. And only constants on right hand side.

\[
\min \mu \\
\forall e: \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i: \sum_{p \in P_i} f_p = D_i \\
f_p \geq 0
\]
Dual.

\[ \min \mu \]

\[ \forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \]

\[ \forall i : \sum_{p \in P_i} f_p = D_i \]

\[ f_p \geq 0 \]
Dual.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p & \geq 0 \quad d_e \\
\forall i : \sum_{p \in P_i} f_p & = D_i \quad d_i \\
f_p & \geq 0
\end{align*}
\]

Introduce variable for each constraint.

Introduce constraint for each var:

\[
\begin{align*}
\mu & \rightarrow \sum e c_e d_e = 1 \\
f_p & \rightarrow \forall p \ni e f_p \geq 0 \\
d_i & \rightarrow \forall i : \sum_{p \in P_i} f_p = D_i \\
\end{align*}
\]

Objective: right hand sides.

\[
\begin{align*}
\max \sum_i D_i d_i \\
\end{align*}
\]

Introduce variable for each constraint.
Dual.

\[
\begin{align*}
\text{min } \mu \\
\forall e: & \mu c_e - \sum_{p \ni e} f_p \geq 0 & d_e \\
\forall i: & \sum_{p \in P_i} f_p = D_i & d_i \\
& f_p \geq 0 \\
\end{align*}
\]

Introduce variable for each constraint. Introduce constraint for each var:
Dual.

\[
\begin{align*}
\min & \mu \\
\forall e & : \mu c_e - \sum_{p \ni e} f_p \geq 0 && d_e \\
\forall i & : \sum_{p \in P_i} f_p = D_i && d_i \\
f_p & \geq 0
\end{align*}
\]

Introduce variable for each constraint. Introduce constraint for each var:
\[
\mu
\]
Dual.

\[
\begin{align*}
\text{min } \mu & \\
\forall e : & \mu c_e - \sum_{p \ni e} f_p \geq 0 & d_e \\
\forall i : & \sum_{p \in P_i} f_p = D_i & d_i \\
& f_p \geq 0
\end{align*}
\]

Introduce variable for each constraint. 
Introduce constraint for each var: 
\[
\mu \rightarrow \sum_e c_e d_e = 1.
\]
Dual.

\[
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : \sum_{p \in P_i} f_p = D_i \\
\forall p : f_p \geq 0
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\mu \rightarrow \sum_e c_e d_e = 1.
\]

Weak duality: toll lower bounds routing.
Strong Duality.
Tight lower bound.
First lecture.
Or Experts.
Complementary Slackness: only route on shortest paths only have toll on congested edges.
Dual.

\[
\begin{align*}
\min & \quad \mu \\
\forall e : & \quad \mu c_e - \sum_{p \ni e} f_p \geq 0 \quad \text{for } d_e \\
\forall i : & \quad \sum_{p \in P_i} f_p = D_i \quad \text{for } d_i \\
\text{subject to } & \quad f_p \geq 0
\end{align*}
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\begin{align*}
\mu & \rightarrow \sum_e c_e d_e = 1. \\
f_p & \rightarrow \forall p \in P_i \ d_i - \sum_{e \in p} d_e \leq 0.
\end{align*}
\]
Dual.

\[
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : \sum_{p \in P_i} f_p = D_i \\
d_e \\
d_i \\
f_p \geq 0
\]

Introduce variable for each constraint. Introduce constraint for each var:

\[\mu \rightarrow \sum e c_e d_e = 1. \quad f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0.\]
Dual.

\[
\begin{align*}
\text{min } \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p &\geq 0 & d_e \\
\forall i : \sum_{p \in P_i} f_p & = D_i & d_i \\
\sum_{\forall e} c_e d_e & = 1 & f_p &\geq 0 \\
\end{align*}
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\begin{align*}
\mu &\rightarrow \sum_e c_e d_e = 1. & f_p &\rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0. \\
\end{align*}
\]
Objective: right hand sides.

Toll problem!
Weak duality: toll lower bounds routing.
Strong Duality.
Tight lower bound.
First lecture.
Or Experts.

Complementary Slackness: only route on shortest paths
only have toll on congested edges.
Dual.

\[ \min \mu \]
\[ \forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \]
\[ \forall i : \sum_{p \in P_i} f_p = D_i \]
\[ f_p \geq 0 \]

Introduce variable for each constraint.
Introduce constraint for each var:
\[ \mu \rightarrow \sum_e c_e d_e = 1. \]
\[ f_p \rightarrow \forall p \in P_i \; d_i - \sum_{e \in p} d_e \leq 0. \]

Objective: right hand sides.
\[ \max \sum_i D_i d_i \]

\[ \forall p \in P_i : d_i \leq \sum_{e \in p} d(e) \]
\[ \sum_e c_e d_e = 1 \]
Dual.

\[
\begin{align*}
\min & \mu \\
\forall e : & \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : & \sum_{p \in P_i} f_p = D_i \\
f_p & \geq 0
\end{align*}
\]

d\_e

d\_i

Introduce variable for each constraint.
Introduce constraint for each var:
\(\mu \rightarrow \sum_e c_e d_e = 1\). \(f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0\).
Objective: right hand sides. \(\max \sum_i D_i d_i\)

\[
\max \sum_i D_i d_i
\]

\(\forall p \in P_i : d_i \leq \sum_{e \in p} d(e)\)

\(\sum_e c_e d_e = 1\)

d\_i \text{- shortest } s_i, t_i \text{ path length.}
\[ \min \mu \]
\[ \forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \]
\[ \forall i : \sum_{p \in P_i} f_p = D_i \]
\[ f_p \geq 0 \]

Introduce variable for each constraint.
Introduce constraint for each var:
\[ \mu \rightarrow \sum_e c_e d_e = 1. \]
\[ f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0. \]

Objective: right hand sides. \( \max \sum_i D_i d_i \)

\[ \max \sum_i D_i d_i \]
\[ \forall p \in P_i : d_i \leq \sum_{e \in p} d(e) \]
\[ \sum_e c_e d_e = 1 \]

\( d_i \) - shortest \( s_i, t_i \) path length. Toll problem!
Dual.

\[
\begin{align*}
\min \mu & \\
\forall e: \mu c_e - \sum_{p \ni e} f_p & \geq 0 & d_e \\
\forall i: \sum_{p \in P_i} f_p & = D_i & d_i \\
f_p & \geq 0
\end{align*}
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\mu \rightarrow \sum_e c_e d_e = 1. \quad f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0.
\]

Objective: right hand sides. \(\max \sum_i D_i d_i\)

\[
\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i \leq \sum_{e \in p} d(e) & \quad \sum_e c_e d_e = 1 \\
d_i - \text{shortest } s_i, t_i \text{ path length. Toll problem!}
\end{align*}
\]

Weak duality: toll lower bounds routing.

Strong Duality.
Tight lower bound.
First lecture.
Or Experts.

Complementary Slackness: only route on shortest paths
only have toll on congested edges.
Dual.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : \sum_{p \in P_i} f_p = D_i \\
f_p \geq 0
\end{align*}
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\mu \rightarrow \sum_e c_e d_e = 1. \quad f_p \rightarrow \forall p \in P_i \quad d_i - \sum_{e \in p} d_e \leq 0.
\]

Objective: right hand sides. \( \max \sum_i D_i d_i \)

\[
\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i \leq \sum_{e \in p} d(e) \\
\sum_e c_e d_e = 1
\end{align*}
\]

\( d_i \) - shortest \( s_i, t_i \) path length. Toll problem!
Weak duality: toll lower bounds routing.
Strong Duality.
Dual.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p & \geq 0 & d_e \\
\forall i : \sum_{p \in P_i} f_p &= D_i & d_i \\
f_p & \geq 0 \\
\end{align*}
\]

Introduce variable for each constraint. Introduce constraint for each var:

\[
\begin{align*}
\mu & \rightarrow \sum_e c_e d_e = 1. & f_p & \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0. \\
\text{Objective: right hand sides.} & \max \sum_i D_i d_i \\
\end{align*}
\]

\[
\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i & \leq \sum_{e \in p} d(e) & \sum_e c_e d_e = 1
\end{align*}
\]

\(d_i\) - shortest \(s_i, t_i\) path length. Toll problem! Weak duality: toll lower bounds routing. Strong Duality. Tight lower bound.
Dual.

\[
\begin{align*}
\min & \quad \mu \\
\forall e : & \quad \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : & \quad \sum_{p \in P_i} f_p = D_i \\
& \quad f_p \geq 0
\end{align*}
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\mu \rightarrow \sum_e c_e d_e = 1. \quad f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0.
\]

Objective: right hand sides. \( \max \sum_i D_i d_i \)

\[
\begin{align*}
\max & \quad \sum_i D_i d_i \\
\forall p \in P_i : & \quad d_i \leq \sum_{e \in p} d(e) \\
& \quad \sum_e c_e d_e = 1
\end{align*}
\]

\( d_i \) - shortest \( s_i, t_i \) path length. Toll problem!
Weak duality: toll lower bounds routing.
Strong Duality. Tight lower bound. First lecture.
Dual.

\[ \min \mu \]
\[ \forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \]
\[ \forall i : \sum_{p \in P_i} f_p = D_i \]
\[ f_p \geq 0 \]

Introduce variable for each constraint.
Introduce constraint for each var:
\[ \mu \rightarrow \sum_e c_e d_e = 1. \]
\[ f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0. \]

Objective: right hand sides. \( \max \sum_i D_i d_i \)

\[ \max \sum_i D_i d_i \]
\[ \forall p \in P_i : d_i \leq \sum_{e \in p} d(e) \]
\[ \sum_e c_e d_e = 1 \]

\( d_i \) - shortest \( s_i, t_i \) path length. Toll problem!

Weak duality: toll lower bounds routing.

Dual.

\[
\begin{align*}
\min \mu \\
\forall e: \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i: \sum_{p \in P_i} f_p = D_i \\
f_p \geq 0
\end{align*}
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\mu \to \sum_e c_e d_e = 1. \quad f_p \to \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0.
\]

Objective: right hand sides.
\[
\max \sum_i D_i d_i
\]

\[
\forall p \in P_i: d_i \leq \sum_{e \in p} d(e) \quad \sum_e c_e d_e = 1
\]

\(d_i\) - shortest \(s_i, t_i\) path length. Toll problem!

Weak duality: toll lower bounds routing.


Complementary Slackness:
Dual.

\[
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : \sum_{p \in P_i} f_p = D_i \\
f_p \geq 0
\]

Introduce variable for each constraint.

Introduce constraint for each var:

\[
\mu \rightarrow \sum_e c_e d_e = 1. \quad f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0.
\]

Objective: right hand sides. \(\max \sum_i D_i d_i\)

\[
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i \leq \sum_{e \in p} d(e) \\
\sum_e c_e d_e = 1
\]

\(d_i\) - shortest \(s_i, t_i\) path length. Toll problem!

Weak duality: toll lower bounds routing.


Complementary Slackness: only route on shortest paths
Dual.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p & \geq 0 \\
\forall i : \sum_{p \in P_i} f_p & = D_i \\
 f_p & \geq 0
\end{align*}
\]

Introduce variable for each constraint.
Introduce constraint for each var:
\[
\mu \rightarrow \sum_e c_e d_e = 1. \quad f_p \rightarrow \forall p \in P_i d_i - \sum_{e \in p} d_e \leq 0.
\]
Objective: right hand sides. \(\max \sum_i D_i d_i\)

\[
\max \sum_i D_i d_i
\]

\[
\forall p \in P_i : d_i \leq \sum_{e \in p} d(e) \quad \sum_e c_e d_e = 1
\]

\(d_i\) - shortest \(s_i, t_i\) path length. Toll problem!

Weak duality: toll lower bounds routing.


Complementary Slackness: only route on shortest paths
only have toll on congested edges.
Matrix View

\( f_p \) variable for path \( e_1, e_2, \ldots, e_k \). \( p \) connects \( s_i, t_i \).

<table>
<thead>
<tr>
<th></th>
<th>( f_p )</th>
<th>( \mu )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>0</td>
</tr>
<tr>
<td>( d_{e_1} )</td>
<td>.</td>
<td>.</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>( d_{e_2} )</td>
<td>.</td>
<td>.</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>( d_{e_k} )</td>
<td>.</td>
<td>.</td>
<td>-1</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>( d_i )</td>
<td>.</td>
<td>1</td>
<td>.</td>
</tr>
<tr>
<td>obj</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Row constraint: \( c_e \mu - \sum_{p \ni e} f_p \geq 0 \).
Matrix View

$f_p$ variable for path $e_1, e_2, \ldots, e_k$. $p$ connects $s_i, t_i$.

<table>
<thead>
<tr>
<th></th>
<th>$f_p$</th>
<th>$\mu$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{e_1}$</td>
<td>$\cdot$</td>
<td>$\cdots$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{e_2}$</td>
<td>$\cdot$</td>
<td>$\cdots$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$d_{e_k}$</td>
<td>$\cdot$</td>
<td>$\cdots$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$d_i$</td>
<td>$\cdot$</td>
<td>$1$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>obj</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Row constraint: $c_e \mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: $d_e$. 
Matrix View

$f_p$ variable for path $e_1, e_2, \ldots, e_k$. $p$ connects $s_i, t_i$.

<table>
<thead>
<tr>
<th></th>
<th>$f_p$</th>
<th>$\mu$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{e_1}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$d_{e_2}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$d_{e_k}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$d_i$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$1$</td>
</tr>
<tr>
<td>obj</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Row constraint: $c_e \mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: $d_e$.

Row constraint: $\sum_{p \in P_i} f_p = D_i$. 
Matrix View

\( f_p \) variable for path \( e_1, e_2, \ldots, e_k \). \( p \) connects \( s_i, t_i \).

<table>
<thead>
<tr>
<th></th>
<th>( f_p )</th>
<th>( \mu )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{e_1} )</td>
<td>( \ldots )</td>
<td>( -1 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( d_{e_2} )</td>
<td>( \ldots )</td>
<td>( -1 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( d_{e_k} )</td>
<td>( \ldots )</td>
<td>( -1 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( d_i )</td>
<td>( \ldots )</td>
<td>( 1 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>obj</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Row constraint: \( c_e \mu - \sum_{p \ni e} f_p \geq 0 \). Row (dual) variable: \( d_e \).

Row constraint: \( \sum_{p \in P_i} f_p = D_i \). Row (dual) variable: \( d_i \).
Matrix View

$f_p$ variable for path $e_1, e_2, \ldots, e_k$. $p$ connects $s_i, t_i$.

<table>
<thead>
<tr>
<th></th>
<th>$f_p$</th>
<th>$\mu$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{e_1}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$d_{e_2}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>$d_{e_k}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
</tr>
<tr>
<td>$d_i$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$1$</td>
</tr>
<tr>
<td>obj</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Row constraint: $c_e \mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: $d_e$.

Row constraint: $\sum_{p \in P_i} f_p = D_i$. Row (dual) variable: $d_i$.

Column variable: $f_p$. 
Matrix View

\( f_p\) variable for path \( e_1, e_2, \ldots, e_k\). \( p\) connects \( s_i, t_i\).

<table>
<thead>
<tr>
<th>( d_{e_1} )</th>
<th>( f_p )</th>
<th>( \mu )</th>
<th>( \text{rhs} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( 0 )</td>
<td>( \cdot )</td>
</tr>
<tr>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
<td>( \cdot )</td>
</tr>
</tbody>
</table>

\( d_{e_1} \) variable:

\( \cdot \) \( \cdot \) \( -1 \) \( \cdot \) \( \cdot \) \( c_{e_1} \) \( 0 \)

\( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( 0 \)

Row constraint: \( c_{e_1} \mu - \sum_{p \ni e} f_p \geq 0\). Row (dual) variable: \( d_e \).

\( d_{e_2} \) variable:

\( \cdot \) \( \cdot \) \( -1 \) \( \cdot \) \( \cdot \) \( c_{e_2} \) \( 0 \)

\( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( 0 \)

Row constraint: \( \sum_{p \ni e} f_p = D_i \). Row (dual) variable: \( d_i \).

\( d_{e_k} \) variable:

\( \cdot \) \( \cdot \) \( -1 \) \( \cdot \) \( \cdot \) \( c_{e_k} \) \( 0 \)

\( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( \cdot \) \( 0 \)

Row constraint: \( c_{e_k} \mu - \sum_{p \ni e} f_p \geq 0\). Row (dual) variable: \( d_e \).

\( d_i \) variable:

\( \cdot \) \( \cdot \) \( 1 \) \( \cdot \) \( \ldots \) \( D_i \)

Row constraint: \( \sum_{p \ni e} f_p = D_i \). Row (dual) variable: \( d_i \).

\( d_i \) variable:

\( \cdot \) \( \cdot \) \( 1 \) \( \cdot \) \( \cdot \) \( \ldots \) \( D_i \)

Row constraint: \( \sum_{p \ni e} f_p = D_i \). Row (dual) variable: \( d_i \).

Column variable: \( f_p \). Column (dual) constraint:
Matrix View

\( f_p \) variable for path \( e_1, e_2, \ldots, e_k \). \( p \) connects \( s_i, t_i \).

<table>
<thead>
<tr>
<th></th>
<th>( f_p )</th>
<th>( \mu )</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{e_1} )</td>
<td>. . . -1 . . ( c_{e_1} ) 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_{e_2} )</td>
<td>. . . -1 . . ( c_{e_2} ) 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_{e_k} )</td>
<td>. . . -1 . . ( c_{e_k} ) 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_i )</td>
<td>. . 1 . . ( D_i )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>obj</td>
<td>1 1 1 1 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Row constraint: \( c_e \mu - \sum_{p \ni e} f_p \geq 0 \). Row (dual) variable: \( d_e \).

Row constraint: \( \sum_{p \in P_i} f_p = D_i \). Row (dual) variable: \( d_i \).

Column variable: \( f_p \). Column (dual) constraint: \( d_i - \sum_{e \in p} d_e \leq 0 \).
Matrix View

$f_p$ variable for path $e_1, e_2, \ldots, e_k$. $p$ connects $s_i, t_i$.

<table>
<thead>
<tr>
<th></th>
<th>$f_p$</th>
<th>$\mu$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{e_1}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{e_2}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_{e_k}$</td>
<td>$\cdot$</td>
<td>$\cdot$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_i$</td>
<td>$\cdot$</td>
<td>$1$</td>
<td>$D_i$</td>
</tr>
<tr>
<td>obj</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Row constraint: $c_e \mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: $d_e$.

Row constraint: $\sum_{p \in P_i} f_p = D_i$. Row (dual) variable: $d_i$.

Column variable: $f_p$. Column (dual) constraint: $d_i - \sum_{e \in p} d_e \leq 0$.

Column variable: $\mu$. 
## Matrix View

$f_p$ variable for path $e_1, e_2, \ldots, e_k$. $p$ connects $s_i, t_i$.

$$
\begin{array}{cccc}
& f_p & \mu & \text{rhs} \\
\hline
d_{e_1} & \cdot & \ldots & -1 & \cdot & \cdots & c_{e_1} & 0 \\
d_{e_2} & \cdot & \ldots & -1 & \cdot & \cdots & c_{e_2} & 0 \\
d_{e_k} & \cdot & \ldots & -1 & \cdot & \cdots & c_{e_k} & 0 \\
d_i & \cdot & 1 & \cdot & \cdots & D_i & 0 \\
\text{obj} & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Row constraint: $c_e \mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: $d_e$.

Row constraint: $\sum_{p \in P_i} f_p = D_i$. Row (dual) variable: $d_i$.

Column variable: $f_p$. Column (dual) constraint: $d_i - \sum_{e \in p} d_e \leq 0$.

Column variable: $\mu$. Column (dual) constraint:
Matrix View

$f_p$ variable for path $e_1, e_2, \ldots, e_k$. $p$ connects $s_i, t_i$.

<table>
<thead>
<tr>
<th></th>
<th>$f_p$</th>
<th>$\mu$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{e_1}$</td>
<td>...</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$d_{e_2}$</td>
<td>...</td>
<td>-1</td>
<td>...</td>
</tr>
<tr>
<td>$d_{e_k}$</td>
<td>...</td>
<td>-1</td>
<td>...</td>
</tr>
<tr>
<td>$d_i$</td>
<td>1</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>obj</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Row constraint: $c_e \mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: $d_e$.

Row constraint: $\sum_{p \in P_i} f_p = D_i$. Row (dual) variable: $d_i$.

Column variable: $f_p$. Column (dual) constraint: $d_i - \sum_{e \in p} d_e \leq 0$.

Column variable: $\mu$. Column (dual) constraint: $\sum_e d(e)c(e) = 1$. 

Exercise: objectives?
**Matrix View**

$f_p$ variable for path $e_1, e_2, \ldots, e_k$. $p$ connects $s_i, t_i$.

<table>
<thead>
<tr>
<th></th>
<th>$f_p$</th>
<th>$\mu$</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{e_1}$</td>
<td>\cdots -1 \cdots</td>
<td>$c_{e_1}$</td>
<td>0</td>
</tr>
<tr>
<td>$d_{e_2}$</td>
<td>\cdots -1 \cdots</td>
<td>$c_{e_2}$</td>
<td>0</td>
</tr>
<tr>
<td>$d_{e_k}$</td>
<td>\cdots -1 \cdots</td>
<td>$c_{e_k}$</td>
<td>0</td>
</tr>
<tr>
<td>$d_i$</td>
<td>\cdots 1 \cdots</td>
<td>$D_i$</td>
<td></td>
</tr>
</tbody>
</table>

Row constraint: $c_e \mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: $d_e$.

Row constraint: $\sum_{p \in P_i} f_p = D_i$. Row (dual) variable: $d_i$.

Column variable: $f_p$. Column (dual) constraint: $d_i - \sum_{e \in p} d_e \leq 0$.

Column variable: $\mu$. Column (dual) constraint: $\sum_e d(e) c(e) = 1$.

Exercise: objectives?
Exponential size.

Multicommodity flow.

\[
\begin{align*}
\text{min } & \mu \\
\forall e : & \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : & \sum_{p \in P_i} f_p = d_i \\
f_p & \geq 0
\end{align*}
\]
Exponential size.  
Multicommodity flow.  

$$\begin{align*}  
\min \mu \\
\forall e : & \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : & \sum_{p \in P_i} f_p = d_i \\
& f_p \geq 0 
\end{align*}$$

Dual is.

$$\begin{align*}  
\max \sum_i D_i d_i \\
\forall p \in P_i : & d_i \leq \sum_{e \in p} d(e) 
\end{align*}$$

Exponential sized programs?  
Answer 1: We solved anyway!  
Answer 2: Ellipsoid algorithm. Find violated constraint $\rightarrow$ poly time algorithm.  
Answer 3: there is polynomial sized formulation.  
Question: what is it?
Exponential size.

Multicommodity flow.

$$\begin{align*}
\min \mu \\
\forall e : & \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : & \sum_{p \in P_i} f_p = d_i \\
f_p & \geq 0
\end{align*}$$

Dual is.

$$\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i & \leq \sum_{e \in p} d(e)
\end{align*}$$

Exponential sized programs?
Exponential size.
Multicommodity flow.

\[
\text{min } \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : \sum_{p \in P_i} f_p = d_i \\
f_p \geq 0
\]

Dual is.

\[
\text{max } \sum_i D_i d_i \\
\forall p \in P_i : d_i \leq \sum_{e \in p} d(e)
\]

Exponential sized programs?

Answer 1:

We solved anyway!

Answer 2:

Ellipsoid algorithm.
Find violated constraint \(\rightarrow\) poly time algorithm.

Answer 3: there is polynomial sized formulation.

Question: what is it?
Exponential size.

Multicommodity flow.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p & \geq 0 \\
\forall i : \sum_{p \in P_i} f_p & = d_i \\
f_p & \geq 0
\end{align*}
\]

Dual is.

\[
\begin{align*}
\max \sum_{i} D_i d_i \\
\forall p \in P_i : d_i & \leq \sum_{e \in p} d(e)
\end{align*}
\]

Exponential sized programs?

Answer 1: We solved anyway!
Exponential size.
Multicommodity flow.

\[ \min \mu \]
\[ \forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \]
\[ \forall i : \sum_{p \in P_i} f_p = d_i \]
\[ f_p \geq 0 \]

Dual is.

\[ \max \sum_i D_i d_i \]
\[ \forall p \in P_i : d_i \leq \sum_{e \in p} d(e) \]

Exponential sized programs?
Answer 1: We solved anyway!
Answer 2:

Ellipsoid algorithm.
Find violated constraint → poly time algorithm.
Answer 3: there is polynomial sized formulation.
Question: what is it?
**Exponential size.**

Multicommodity flow.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p & \geq 0 \\
\forall i : \sum_{p \in P_i} f_p &= d_i \\
f_p & \geq 0
\end{align*}
\]

Dual is.

\[
\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i & \leq \sum_{e \in p} d(e)
\end{align*}
\]

Exponential sized programs?

Answer 1: We solved anyway!

Answer 2: Ellipsoid algorithm.
Exponential size.

Multicommodity flow.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0 \\
\forall i : \sum_{p \in P_i} f_p = d_i \\
f_p \geq 0
\end{align*}
\]

Dual is.

\[
\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i \leq \sum_{e \in p} d(e)
\end{align*}
\]

Exponential sized programs?

Answer 1: We solved anyway!

Answer 2: Ellipsoid algorithm.

Find violated constraint → poly time algorithm.
Exponential size.

Multicommodity flow.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p & \geq 0 \\
\forall i : \sum_{p \in P_i} f_p & = d_i \\
f_p & \geq 0
\end{align*}
\]

Dual is.

\[
\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i & \leq \sum_{e \in p} d(e)
\end{align*}
\]

Exponential sized programs?

Answer 1: We solved anyway!

Answer 2: Ellipsoid algorithm. Find violated constraint $\rightarrow$ poly time algorithm.

Answer 3: there is polynomial sized formulation.
Exponential size.
Multicommodity flow.

\[
\begin{align*}
\min \mu \\
\forall e : \mu c_e - \sum_{p \ni e} f_p &\geq 0 \\
\forall i : \sum_{p \in P_i} f_p &= d_i \\
f_p &\geq 0
\end{align*}
\]

Dual is.

\[
\begin{align*}
\max \sum_i D_i d_i \\
\forall p \in P_i : d_i &\leq \sum_{e \in p} d(e)
\end{align*}
\]

Exponential sized programs?

Answer 1: We solved anyway!

Answer 2: Ellipsoid algorithm.
Find violated constraint \(\rightarrow\) poly time algorithm.

Answer 3: there is polynomial sized formulation.
Question: what is it?
See you on Thursday.