Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.

Garden has 40 yards of sunny land and 75 yards of shady land.

To pea or not to pea, that is the question!

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.
x_1- to pea! x_2 to carrot ?
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.
$2x_1 \leq 40$
Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.
$3x_2 \leq 75$
Can't make negative! $x_1, x_2 \geq 0$.
A linear program.

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

Optimal point?

Try every point if we only had time!

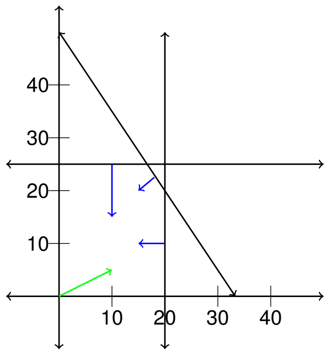
How many points?

Real numbers?

Infinite. Uncountably infinite!

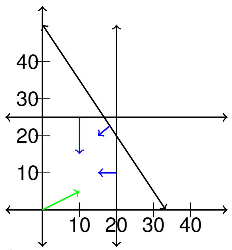
A linear program.

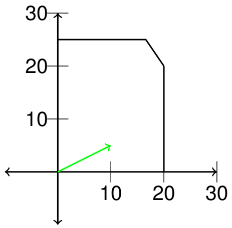
$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$



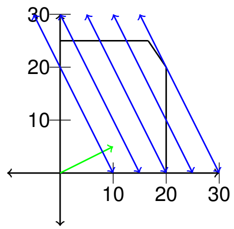
Optimal point?

Feasible Region.

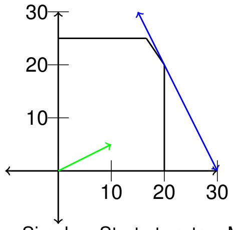




Convex.
Any two points in region connected by a line in region.
Algebraically:
If x and x' satisfy constraint: $ax \leq b$ and $ax' \leq b$,
$x'' = \alpha x + (1-\alpha)x' \rightarrow ax'' \leq b$.



Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if m constraints and 2 variables.
For n variables, m constraints, how many?
nm? $\binom{m}{n}$? $n+m$?
$\binom{m}{n}$
Finite!!!!!!
Exponential in the number of variables.



$\max 4x_1 + 2x_2$
$2x_1 \leq 40$
$3x_2 \leq 75$
$3x_1 + 2x_2 \leq 100$
$x_1, x_2 \geq 0$

Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. This example.
(0,0) objective 0. $\rightarrow (0,25)$ objective 50.
$\rightarrow (16\frac{2}{3}, 25)$ objective $115\frac{2}{3}$
$\rightarrow (20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/2$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Objective value: 120.
Can we do better? No!
Dual problem: add equations to get best upper bound.

Duality.

$$\max x_1 + 8x_2$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..
....so $x_1 + 8x_2 \leq 4 + 8(3) = 28$.
Added equation 1 and 8 times equation 2
yields bound on objective..
Better solution?
Better upper bound?

Duality.

$$\max x_1 + 8x_2$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
$x_1 + 8x_2 \leq 6(3) + 7 = 25$.

Thus, the value is at most 25.

The upper bound is same as solution!

Proof of optimality!

Duality:example

Idea: Add up positive linear combination of inequalities to "get" upper bound on optimization function.

Will this always work?

How to find best upper bound?

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...
$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$
The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

Find best y_i's to minimize upper bound?

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

$$\min 4y_1 + 3y_2 + 7y_3$$
$$y_1 + y_3 \geq 1$$
$$y_2 + 2y_3 \geq 8$$
$$y_1, y_2, y_3 \geq 0$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1,3)$; Dual: $(y_1, y_2, y_3) = (0,6,1)$.

Value of both is 25!

Primal is optimal ... and dual is optimal!

The dual.

In general.

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual (D)

Feasible (x, y)
$P(x) = c \cdot x \leq y^T Ax \leq y^T b \cdot x = D(y)$.

Strong Duality: next lecture, previous lectures maybe?

Complementary Slackness

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Given A, b, c, and feasible solutions x and y.

Solutions x and y are both optimal if and only if
$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

$x_i(c_i - (y^T A)_i) = 0 \rightarrow$
$\sum_i (c_i - (y^T A)_i) x_i = cx - y^T Ax \rightarrow cx = y^T Ax$.

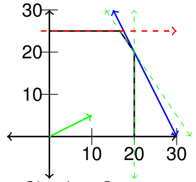
$y_j(b_j - (Ax)_j) = 0 \rightarrow$
$\sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \rightarrow by = y^T Ax$.

$cx = by$.

If both are feasible, $cx \leq by$, so must be optimal.

In words: nonzero dual variables only for tight constraints!

Again: simplex



$\max 4x_1 + 2x_2$
$3x_1 \leq 60$
$3x_2 \leq 75$
$3x_1 + 2x_2 \leq 100$
$x_1, x_2 \geq 0$

Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Geometrically and Complementary slackness:
Add tight constraints to "dominate objective function."
Don't add this equation! Shifts.

Example: review.

$\max x_1 + 8x_2$	$\min 4y_1 + 3y_2 + 7y_3$
$x_1 \leq 4$	$y_1 + y_3 \geq 1$
$x_2 \leq 3$	$y_2 + 2y_3 \geq 8$
$x_1 + 2x_2 \leq 7$	$x_1, x_2 \geq 0$
$y_1, y_2, y_3 \geq 0$	

"Matrix form"

$\max [1, 8] \cdot [x_1, x_2]$ $\qquad$ $\min [4, 3, 7] \cdot [y_1, y_2, y_3]$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \qquad [y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$[x_1, x_2] \geq 0$ $\qquad$ $[y_1, y_2, y_3] \geq 0$

Matrix equations.

$\max [1, 8] \cdot [x_1, x_2]$ $\qquad$ $\min [4, 3, 7] \cdot [y_1, y_2, y_3]$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \qquad [y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$[x_1, x_2] \geq 0$ $\qquad$ $[y_1, y_2, y_3] \geq 0$

We can rewrite the above in matrix form.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \qquad c = [1, 8] \qquad b = [4, 3, 7]$$

The primal is $Ax \leq b, \max c \cdot x, x \geq 0$.
The dual is $y^T A \geq c, \min b \cdot y, y \geq 0$.

Rules for School...

or..."Rules for taking duals"
Canonical Form.

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Standard:

$Ax \leq b, \max cx, x \geq 0 \leftrightarrow y^T A \geq c, \min by, y \geq 0$.

min $\leftrightarrow$ max

$\geq \leftrightarrow \leq$

"inequalities" $\leftrightarrow$ "nonnegative variables"

"nonnegative variables" $\leftrightarrow$ "inequalities"

Another useful trick: Equality constraints. "equalities" $\leftrightarrow$ "unrestricted variables."

Maximum Weight Matching.

Bipartite Graph $G=(V,E)$, $w:E\to Z$.
Find maximum weight perfect matching.
Solution: x_e indicates whether edge e is in matching.

$$\max \sum_e w_e x_e$$
$$\forall v: \sum_{e=(u,v)} x_e = 1 \qquad p_v$$
$$x_e \geq 0$$

Dual.
Variable for each constraint. p_v unrestricted.
Constraint for each variable. Edge e, $p_u + p_v \geq w_e$
Objective function from right hand side. $\min \sum_v p_v$

$$\min \sum_v p_v$$
$$\forall e=(u,v): \quad p_u + p_v \geq w_e$$

Weak duality? Price function upper bounds matching.
$\sum_{e\in M} w_e x_e \leq \sum_{e=(u,v)\in M} p_u + p_v \leq \sum_v p_u$.

Strong Duality? Same value solutions. Hungarian algorithm !!!

Matrix View.

x_e variable for $e=(u,v)$.

			x_e		rhs
	·	···	0	···	1
	⋮	⋮	⋮	⋮	1
p_u	·	···	1	···	1
	·	···	0	···	1
	⋮	⋮	⋮	⋮	1
	·	···	0	···	1
p_v	·	···	1	···	1
	·	···	0	···	1
	⋮	⋮	⋮	⋮	1
obj	·	·	w_e	·	

Row equation: $\sum_{e=(u,v)} x_e = 1$. Row (dual) variable: p_u.

Column variable: x_e. Column (dual) constraint: $p_u + p_v \geq 1$.

Exercise: objectives?

Complementary Slackness.

$$\max \sum_e w_e x_e$$
$$\forall v: \sum_{e=(u,v)} x_e = 1 \qquad p_v$$
$$x_e \geq 0$$

Dual:

$$\min \sum_v p_v$$
$$\forall e=(u,v): \quad p_u + p_v \geq w_e$$

Complementary slackness:
Only match on tight edges.
Nonzero p_u on matched u.

Constraint for u:
$\sum_{e=(u,v)} x_e \leq 1$.
Nonzero p_u on matched u.

Multicommodity Flow.

Given $G=(V,E)$, and capacity function $c:E\to Z$, and pairs $(s_1,t_1),\ldots,(s_k,t_k)$ with demands $D_1,\ldots,D_k$.
Route D_i flow for each s_i,t_i pair,
so every edge has $\leq \mu c(e)$ flow
with minimum μ.

variables: f_p flow on path p.
P_i -set of paths with endpoints s_i,t_i.

$$\min \mu$$
$$\forall e: \sum_{p\ni e} f_p \leq \mu c_e$$
$$\forall i: \sum_{p\in P_i} f_p = D_i$$
$$f_p \geq 0$$

Take the dual.

$$\min \mu$$
$$\forall e: \sum_{p\ni e} f_p \leq \mu c_e$$
$$\forall i: \sum_{p\in P_i} f_p = D_i$$
$$f_p \geq 0$$

Modify to make it $\geq$, which "go with min.
And only constants on right hand side.

$$\min \mu$$
$$\forall e: \mu c_e - \sum_{p\ni e} f_p \geq 0$$
$$\forall i: \sum_{p\in P_i} f_p = D_i$$
$$f_p \geq 0$$

Dual.

$$\min \mu$$
$$\forall e: \mu c_e - \sum_{p\ni e} f_p \geq 0 \qquad d_e$$
$$\forall i: \sum_{p\in P_i} f_p = D_i \qquad d_i$$
$$f_p \geq 0$$

Introduce variable for each constraint.
Introduce constraint for each var:
$\mu \ \to \sum_e c_e d_e = 1$. $f_p \ \to \forall p\in P_i \ d_i - \sum_{e\in p} d_e \leq 0$.
Objective: right hand sides. $\max \sum_i D_i d_i$

$$\max \sum_i D_i d_i$$
$$\forall p\in P_i: d_i \leq \sum_{e\in p} d(e) \qquad \sum_e c_e d_e = 1$$

d_i - shortest s_i,t_i path length. Toll problem!
Weak duality: toll lower bounds routing.
Strong Duality. Tight lower bound. First lecture. Or Experts.
Complementary Slackness: only route on shortest paths
only have toll on congested edges.

Matrix View

f_p variable for path $e_1, e_2, \ldots, e_k$. p connects s_i, t_i.

			f_p		μ	rhs
	$\cdot$	$\cdots$	0	$\cdots$	$\cdot$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	0
d_{e_1}	$\cdot$	$\cdots$	-1	$\cdots$	c_{e_1}	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	0
d_{e_2}	$\cdot$	$\cdots$	-1	$\cdots$	c_{e_2}	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	0
d_{e_k}	$\cdot$	$\cdots$	-1	$\cdots$	c_{e_k}	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	0
d_i	$\cdot$	$\cdot$	1	$\cdot$	$\cdots$	D_i
obj	1	1	1	1		

Row constraint: $c_e\mu - \sum_{p \ni e} f_p \geq 0$. Row (dual) variable: d_e.

Row constraint: $\sum_{p \in P_i} f_p = D_i$. Row (dual) variable: d_i.

Column variable: f_p. Column (dual) constraint: $d_i - \sum_{e \in p} d_e \leq 0$.

Column variable: μ. Column (dual) constraint: $\sum_e d(e)c(e) = 1$.

Exercise: objectives?

Exponential size.

Multicommodity flow.

$$\min \mu$$
$$\forall e : \mu c_e - \sum_{p \ni e} f_p \geq 0$$
$$\forall i : \sum_{p \in P_i} f_p = d_i$$
$$f_p \geq 0$$

Dual is.

$$\max \sum_i D_i d_i$$
$$\forall p \in P_i : d_i \leq \sum_{e \in p} d(e)$$

Exponential sized programs?

Answer 1: We solved anyway!

Answer 2: Ellipsoid algorithm.
Find violated constraint $\rightarrow$ poly time algorithm.

Answer 3: there is polynomial sized formulation.
Question: what is it?

See you on Thursday.