Last Time:

Path Routing Problem. (Min)
Toll Problem. (Max)
Toll $\leq$ Path.
Algs: Exp. Weights for Tolls/Shortest Paths for Path.

"Near" optimal solution
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CS270: Lecture 3.

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“Near” optimal solution $s$!
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Today: continuous view.
CS270: Lecture 3.

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- Path Routing Problem. (Min)
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Toll ≤ Path.

Algs: Exp. Weights for Tolls/Shortest Paths for Path.

“Near” optimal solution

Today: continuous view.

And: Strategic Games
Gradient Descent.

Give differentiable $f(x)$, find minimum.
Gradient Descent.

Give differentiable \( f(x) \), find minimum.

Alg:
   While “not good enough”:
      \[ x^{i+1} = x^i - \varepsilon_i \nabla f(x^i). \]
Gradient Descent.

Give differentiable $f(x)$, find minimum.

Alg:

\[
\text{While "not good enough":} \\
x^{i+1} = x^i - \varepsilon_i \nabla f(x^i).
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\nabla (f(x^i)) = 0 \implies \text{Optimal.}
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Constrained: project gradient into affine space.
Gradient Descent.

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Constrained: project gradient into affine space.

Projected($\nabla (f(x^i)) = 0 \implies \text{Optimal.}$

Dumber: just move to $x^{(i+1)}$ with smaller $f(x^{(i)})$ in affine subspace.
Routing and Function minimization.

Simple Version of Routing problem.

Route $X$ units of flow between $s$ and $t$.

Minimize congestion.

$$\min \max e \cdot c(e).$$

Not smooth.

Smoothing functions: minimize max

$$\max e \cdot c(e).$$

$$f(R) = \sum e^2 \cdot c(e).$$

$$f'(R) = \sum e \cdot c(e)^2.$$

Good smoothing?

Thm: Routing $R$ that minimizes $f(R)$ has max $e \cdot c(e) = c(R) \leq c_{opt} + \log m$.

Proof:

Max Congestion Optimal routing, $R^*$, has $f(R^*) \leq m^2 \cdot c_{opt}$.

Why?

$m$ edges each with congestion at most $c_{opt}$.

This routing has $f(R) \geq 2 \cdot c(R)$.

$\rightarrow m^2 \cdot c_{opt} \geq f(R) \geq 2 \cdot c(R)$.

$\rightarrow c_{opt} + \log m \geq c(R)$.
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Proof:
Max Congestion Optimal routing, $R^*$, has $f(R^*) \leq m2^{c_{opt}}$. 
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Why? \( m \) edges each with congestion at most \( c_{opt} \).
This routing has \( f(R) \geq 2^{c(R)} \).
Routing and Function minimization.

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Proof:
Max Congestion Optimal routing, $R^*$, has $f(R^*) \leq m^{2c_{opt}}$.
Why? $m$ edges each with congestion at most $c_{opt}$.
This routing has $f(R) \geq 2^{c(R)}$.

$\rightarrow m^{2c_{opt}} \geq f(R) \geq 2^{c(R)}$.
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Optimization Setup: continued.

$R$ “routes” $F$ units of flow for one pair $(s, t)$. 

$\nabla f(R) = c'(e) \log_2 c(e)$. With respect to what? What are the variables? What choices do we have?
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As optimization: continued

$R$ “routes” a unit flow for one pair $(s, t)$.
As optimization: continued

\( R \) “routes” a unit flow for one pair \((s, t)\).
“Decision Variable”.

\[
x(p) \text{ flow along } p.
\]

Constraint: sum of \( x(p) \) is 1.

What is \( c(e) \) in terms of \( x(p) \)?

\[
A[e, p] = 1 \text{ if } e \in p \text{ and } 0 \text{ otherwise}.
\]

Now, we have:

\[
c = Ax, \text{ minimize max } c(e) \text{ where } \sum_p x(p) = 1.
\]
As optimization: continued

\( R \) “routes” a unit flow for one pair \((s, t)\).

“Decision Variable”.
For an \( s - t \) path \( p \), \( x(p) \) flow along \( p \).
As optimization: continued

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Exponential number!
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\[ c = Ax, \quad \text{minimize} \max_e c(e) \quad \text{where} \sum_p x(p) = 1. \]
...and smoothing: continued.

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\[ c = Ax, \quad \text{minimize } \max_e c(e) \quad \text{where } \sum_p x(p) = F. \]
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Smooth version: minimize \( \sum_e 2^{c(e)}. \)
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Minimum gives solution within additive \( \log m \) of optimal.

Better?: \( F \) to \( 2F \)
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Algorithm: reduce potential!
...and smoothing: continued.

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Algorithm: reduce potential! \( \sum_e 2^{c(e)} \).

Best possible: a factor of two off.
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Oscillates if move when length of path not smaller by factor of 2.
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Oscillates if move when length of path not smaller by factor of 2.
\[ \sum_e 2^{c(e)} \to \sum_e (1 + \varepsilon)^{c(e)}. \]
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Approximate Equilibrium: \( (1 + 2\varepsilon)C_{opt} + \delta \log n/\varepsilon. \)
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Approximate Equilibrium: \( (1 + 2\varepsilon)C_{opt} + \delta \log n/\varepsilon \).

Convergence time:

Potential drop: \( \geq \varepsilon \sum_{e \in p} 2^{c(e)} \)

Move Size: \( \delta \).
...and smoothing: continued.

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Minimum gives solution within additive \( \log m \) of optimal.

Better?: \( F \) to \( 2F \) \( \implies \) error divides by two.

\( F \) to \( F/\delta \) \( \implies \) additive error is \( \delta \log m \).

Algorithm: reduce potential! \( \sum_e 2^{c(e)} \).

Best possible: a factor of two off.

Oscillates if move when length of path not smaller by factor of 2.

\[ \sum_e 2^{c(e)} \to \sum_e (1 + \varepsilon)^{c(e)}. \]

Approximate Equilibrium: \( (1 + 2\varepsilon) C_{opt} + \delta \log n/\varepsilon \).

Convergence time:

Potential drop: \( \geq \varepsilon \sum_{e \in p} 2^{c(e)} \)

Move Size: \( \delta \).

Time: \( \text{Poly}(1/\varepsilon, 1/\delta, n, m) \).
Continuous view: calculus.

\[ c = Ax, \quad \text{minimize } \max_e c(e) \quad \text{where } \sum_p x(p) = F. \]
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\[ A[e, p] - 1 \text{ if } e \in p, \ 0 \text{ otherwise.} \]
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c = Ax, minimize \( \max_e c(e) \) where \( \sum_p x(p) = F \).

\( A[e, p] = 1 \) if \( e \in p \), 0 otherwise.

c is indexed by \( e \) or has dimension \( m \).
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\[ \implies \nabla_x (f(R)) \propto A^t \overrightarrow{2^{c(e)}}. \]
Projection.

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We also have: \( \sum_p x(p) = F \)

Affine subspace: so can project!
\[ c = Ax \]

e space isocline.

\[ c(e_2) \]

\[ c(e_1) \]

\[ \leftarrow A = I \rightarrow \]

x space feasibility.

\[ x(p_2) \]

\[ F \]

\[ x(p_1) \]
Strategic Games.

$N$ players.
Strategic Games.

$N$ players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$. 
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Each player has strategy set. \{$(S_1, \ldots, S_N)$\}.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \mathbb{R}^N$).
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Example:
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2 players

Player 1: \{ Defect, Cooperate \}.

Player 2: \{ Defect, Cooperate \}.
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What is the best thing for the players to do?
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Both cooperate. Payoff $\langle 3, 3 \rangle$. 

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If player 1 wants to do better, what does she do?
Defects! Payoff $\langle 5, 0 \rangle$.

What does player 2 do now?
Defects! Payoff $\langle .1, .1 \rangle$.

Stable now!

Nash Equilibrium: neither player has incentive to change strategy.
What is the best thing for the players to do?
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What does player 2 do now?
Defects! Payoff (.1, .1)
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What is the best thing for the players to do?

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Lots of interesting Game Theory!

Today: simpler version.
Two Person Zero Sum Games

2 players.

Each player has strategy set:
m strategies for player 1
n strategies for player 2

Payoff function:
$$u(i, j) = (-a, a)$$
(or just $a$).

"Player 1 pays $a$ to player 2."

Zero Sum: Payoff for any pair of strategies sums to 0.

Payoffs by $m$ by $n$ matrix: $A$.

Row player minimizes, column player maximizes.

Roshambo: rock, paper, scissors.

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Any Nash Equilibrium? 

- $(R, R)$?
- $(R, P)$?
- $(R, S)$?

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\begin{array}{ccc}
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Any Nash Equilibrium?

Mixed Strategies.

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How do you play?

Player 1: play each strategy with equal probability.

Player 2: play each strategy with equal probability.

Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.
Mixed Strategies.

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Payoffs?

Can't just look it up in matrix!

Average Payoff.

Expected Payoff.

Sample space: $\Omega = \{(i, j) : i, j \in [1, \ldots, 3]\}$

Random variable $X$ (payoff).

$$E[X] = \sum_{(i, j)} X(i, j) \Pr[(i, j)]$$

Each player chooses independently:

$$\Pr[(i, j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

$$E[X] = \frac{1}{9} \sum_{(i, j)} X(i, j) = 0$$

Payoff for other player?

One payoff!

- Row minimizes.
- Column maximizes.
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

Average Payoff. **Expected Payoff.**
Payoffs: Equilibrium.

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Average Payoff. Expected Payoff.

Sample space: $\Omega = \{ (i,j) : i,j \in [1,..,3] \}$
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

Average Payoff. **Expected Payoff**.

Sample space: \( \Omega = \{(i,j) : i,j \in [1,\ldots,3]\} \)

Random variable \( X \) (payoff).
Payoffs: Equilibrium.

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Sample space: $\Omega = \{(i, j) : i, j \in [1, \ldots, 3]\}$

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$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)]$$
Payoffs: Equilibrium.

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Sample space: \( \Omega = \{(i,j) : i,j \in [1,\ldots,3]\} \)

Random variable \( X \) (payoff).

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Each player chooses independently:
## Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!

**Average Payoff. Expected Payoff.**

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$

Random variable $X$ (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$. 
### Payoffs: Equilibrium.

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**Average Payoff. Expected Payoff.**

Sample space: $$\Omega = \{(i, j) : i, j \in [1, \ldots, 3]\}$$

Random variable $$X$$ (payoff).

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$$E[X]$$
Payoffs: Equilibrium.

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$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j)$$
Payoffs: Equilibrium.

\[ \begin{array}{ccc}
R & P & S \\
\hline
R & .33 & .33 & .33 \\
P & .33 & .33 & .33 \\
S & .33 & .33 & .33 \\
\end{array} \]

Payoffs? Can’t just look it up in matrix!.

Average Payoff. **Expected Payoff**.

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Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!

Average Payoff. **Expected Payoff.**

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Payoff for other player?
Payoffs: Equilibrium.

\[
\begin{array}{ccc}
R & P & S \\
\hline
R & .33 & .33 & .33 \\
\hline
P & .33 & -1 & 0 & 1 \\
\hline
S & .33 & 1 & -1 & 0 \\
\end{array}
\]

Payoffs? Can’t just look it up in matrix!

Average Payoff. **Expected Payoff.**

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Payoff for other player? One payoff!
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!

Average Payoff. **Expected Payoff.**

Sample space: \( \Omega = \{(i,j) : i,j \in [1,..,3]\} \)

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Payoff for other player? One payoff!
- row minimizes.
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!

Average Payoff. **Expected Payoff.**

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\[
E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j) = 0.
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Payoff for other player? One payoff!
- row minimizes. column maximizes.
Will Player 1 change strategy?

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Will Player 1 change strategy? Mixed strategies uncountable!
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock?

Mixed strategy payoff is weighted average of payoffs of pure strategies.

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change!

Same for player 2.

Equilibrium!

Satish Rao (UC Berkeley)
## Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).
### Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

Expected payoff of Paper?
**Equilibrium**

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0. \)

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \[ \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0. \]

Expected payoff of Paper? \[ \frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0. \]

Expected payoff of Scissors?
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

- Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).
- Expected payoff of Paper? \( \frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0 \).
- Expected payoff of Scissors? \( \frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0 \).
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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

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Expected payoff of Scissors? \( \frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0 \).

No better pure strategy.
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.

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No better pure strategy. $\implies$ No better mixed strategy!
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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.

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Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.
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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j])X(i,j)$
Equilibrium

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Equilibrium

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Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times (-1) = 0$.

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Mixed strategy can’t be better than the best pure strategy.
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

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Player 1 has no incentive to change!
Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock?
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Mixed strategy can’t be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

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\[
E[X] = \sum (i,j)(Pr[i] \times Pr[j])X(i,j) = \sum_i Pr[i](\sum_j Pr[j] \times X(i,j))
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Mixed strategy can’t be better than the best pure strategy.
Player 1 has no incentive to change! Same for player 2.

Equilibrium!
Another example plus notation.

Rock, Paper, Scissors, prEempt.

Payoffs.

\[
\begin{array}{cccc}
R & P & S & E \\
\hline
R & 0 & 1 & -1 & 1 \\
P & -1 & 0 & 1 & 1 \\
S & 1 & -1 & 0 & 1 \\
E & -1 & -1 & -1 & 0 \\
\end{array}
\]

Equilibrium? \((E,E)\).

Pure strategy equilibrium.

Notation:

Rock is 1, Paper is 2, Scissors is 3, prEempt is 4.
Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.

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Rock, Paper, Scissors, prEempt.
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Equilibrium? \((E,E)\). Pure strategy equilibrium.

Notation:
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Equilibrium? \((E,E)\). Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
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Rock, Paper, Scissors, prEempt.
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Payoffs.

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Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.

Payoff Matrix.

\[
A = \begin{bmatrix}
0 & 1 & -1 & 1 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & 1 \\
-1 & -1 & -1 & 0 \\
\end{bmatrix}
\]
Playing the boss...

Row has extra strategy: Cheat.
Playing the boss...

Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Playing the boss...

Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

\[
\begin{pmatrix}
0 & 1 & -1 & -1 \\
-1 & 0 & 1 & -1 \\
-1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
\end{pmatrix}
\]

Note: column knows row cheats.

Why play?
Row is column's advisor. ... boss.
Playing the boss...

Row has extra strategy: Cheat.
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Payoff matrix:
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... boss.
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
  0 & 1 & -1 \\
 -1 & 0 & 1 \\
 1 & -1 & 0 \\
 0 & 0 & -1 \\
\end{bmatrix} \]

Equilibrium:
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\).
Equilibrium: play the boss...

\[
A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
\]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).
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Payoff?
Equilibrium: play the boss...

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Equilibrium: play the boss...

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Row Player.
Equilibrium: play the boss...

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A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1\)
Equilibrium: play the boss...

\[ A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)
Strategy 2: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1\)
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
0 & 1 & -1 \\
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\end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)

Strategy 2: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\)
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
  0 & 1 & -1 \\
  -1 & 0 & 1 \\
  1 & -1 & 0 \\
  0 & 0 & -1
\end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)
Strategy 2: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\)
Strategy 3: \(\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0\)
Equilibrium: play the boss...

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A = \begin{bmatrix}
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Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)

Strategy 2: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\)

Strategy 3: \(\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}\)
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


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Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)

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Strategy 4: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1\)
Equilibrium: play the boss...

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Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


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Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)
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Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \[\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\]
Strategy 2: \[\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\]
Strategy 3: \[\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}\]
Strategy 4: \[\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = -\frac{1}{6}\]
**Equilibrium: play the boss...**

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A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


**Row Player.**

**Strategy 1:** \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)

**Strategy 2:** \(\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\)

**Strategy 3:** \(\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}\)

**Strategy 4:** \(\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = -\frac{1}{6}\)

Payoff is \(0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6})\)
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
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\end{bmatrix} \]

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times (-1) = \frac{1}{3}\)
Strategy 2: \(\frac{1}{3} \times (-1) + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\)
Strategy 3: \(\frac{1}{3} \times 1 + \frac{1}{2} \times (-1) + \frac{1}{6} \times 0 = -\frac{1}{6}\)
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Payoff is \(0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6}) = -\frac{1}{6}\)
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Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)

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Payoff is \(0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6}) = -\frac{1}{6}\)

Column player: every column payoff is \(-\frac{1}{6}\).
Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium: Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)
Strategy 2: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\)
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Both only play optimal strategies!
Equilibrium: play the boss...

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  0 & 1 & -1 \\
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 1 & -1 & 0 \\
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Both only play optimal strategies! Complementary slackness.
Equilibrium: play the boss...

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Column player: every column payoff is \(-\frac{1}{6}\).

Both only play optimal strategies! Complementary slackness. Why play more than one?
Equilibrium: play the boss...

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A = \begin{bmatrix}
0 & 1 & -1 \\
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1 & -1 & 0 \\
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\end{bmatrix}
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Column player: every column payoff is \(-\frac{1}{6}\).

Both only play optimal strategies! Complementary slackness.

Why play more than one? Limit opponent payoff!
Two person zero sum games.

$m \times n$ payoff matrix $A$.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.

Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair $(x, y)$:

$$p(x, y) = x^t Ay$$

That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j} = \sum_{i} x_i (\sum_{j} a_{i,j} y_j) = \sum_{i} \sum_{j} x_i a_{i,j} y_j = \sum_{j} (\sum_{i} x_i a_{i,j}) y_j.$$ 

Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$$x^* y^* t Ay = \min_{x} x^t Ay = \max_{y} (x^* y) t Ay$$

(No better column strategy, no better row strategy.)
Two person zero sum games.

$m \times n$ payoff matrix $A$.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Two person zero sum games.

$m \times n$ payoff matrix $A$.

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Payoff for strategy pair $(x, y)$:

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That is,

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Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$$x^t A y^* = \max_y (x^t A y) = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)
Two person zero sum games.

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That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j}$$
Two person zero sum games.

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Payoff for strategy pair $(x, y)$:

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That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j} = \sum_{i} x_i \left( \sum_{j} a_{i,j} y_j \right)$$
Two person zero sum games.

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Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$(x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*$.

(No better column strategy, no better row strategy.)
Two person zero sum games.

$m \times n$ payoff matrix $A$.

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Equilibrium pair: $(x^*, y^*)$?
Two person zero sum games.

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Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$$(x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.$$
Two person zero sum games.

$m \times n$ payoff matrix $A$.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair $(x,y)$:

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$$(x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.$$  

(No better column strategy,
Two person zero sum games.

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Payoff for strategy pair $(x, y)$:

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That is,

$$\sum_{i,j}(x_iy_j)a_{i,j} = \sum_i x_i \left( \sum_j a_{i,j}y_j \right) = \sum_i \sum_j x_i a_{i,j}y_j = \sum_j \left( \sum_i x_i a_{i,j} \right) y_j.$$

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Equilibrium pair: $(x^*, y^*)$?

$$(x^*)^tAy^* = \max_y (x^*)^tAy = \min_x x^tAy^*.$$  
(No better column strategy, no better row strategy.)
Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*. \]

(No better column strategy, no better row strategy.)

---

\(^1\) \(A^{(i)}\) is \(i\)th row.
Equilibrium.

Equilibrium pair: \((x^*, y^*)\)?

\[
p(x, y) = (x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.
\]

(No better column strategy, no better row strategy.)

No row is better:

\[
\min_i A^{(i)} \cdot y = (x^*)^t Ay^*. \quad ^1
\]

\[
^1 A^{(i)} \text{ is } i\text{th row.}
\]
Equilibrium.

Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay*. \]

(No better column strategy, no better row strategy.)

No row is better:

\[ \min_i A^{(i)} \cdot y = (x^*)^t Ay^* . \]

No column is better:

\[ \max_j (A^t)^{(j)} \cdot x = (x^*)^t Ay^*. \]

\[ A^{(i)} \] is \(i\)th row.
Column goes first:

Find $y$, where best row is not too low.

$$ R = \max_y \min_x (x^t A y) $$

Note: $x$ can be $(0,0,\ldots,1,\ldots)$. Example: Roshambo.

Row goes first:

Find $x$, where best column is not high.

$$ C = \min_x \max_y (x^t A y) $$

Again: $y$ of form $(0,0,\ldots,1,\ldots)$. From Texas. Example: Roshambo.

Value of $C$?
Best Response

**Column goes first:**
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots)$.

Example: Roshambo.

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$

Again: $y$ of form $(0, 0, \ldots, 1, \ldots)$.

From Texas.

Example: Roshambo.

**Value of $R$?**
Best Response

**Column goes first:**
Find $y$, where best row is not too low..

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.
Best Response

**Column goes first:**
Find $y$, where best row is not too low.

$$ R = \max_y \min_x (x^t A y). $$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo.
Best Response

Column goes first:
Find $y$, where best row is not too low..

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0,0,\ldots,1,\ldots,0)$.
Example: Roshambo. Value of $R$?
Best Response

**Column goes first:**
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of $R$?

**Row goes first:**
Find $x$, where best column is not high.
Best Response

Column goes first:
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of $R$?

Row goes first:
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$
Best Response

**Column goes first:**
Find $y$, where best row is not too low..

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Note: $x$ can be $(0,0,\ldots,1,\ldots0)$.

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**Row goes first:**
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$$C = \min_x \max_y (x^tAy).$$

Agin: $y$ of form $(0,0,\ldots,1,\ldots0)$.
**Best Response**

**Column goes first:**
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t A y).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of $R$?

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t A y).$$

Again: $y$ of form $(0, 0, \ldots, 1, \ldots 0)$. From Texas.
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Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of $R$?

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$

Again: $y$ of form $(0, 0, \ldots, 1, \ldots 0)$. From Texas.

Example: Roshambo.
Best Response

**Column goes first:**
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0,0,\ldots,1,\ldots0)$.

Example: Roshambo. Value of $R$?

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$

Again: $y$ of form $(0,0,\ldots,1,\ldots0)$. From Texas.

Example: Roshambo. Value of $C$?
Duality.

\[ R = \max_y \min_x (x^t Ay). \]

Weak Duality: \( R \leq C \).

Proof: Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v^* \):
- Row payoffs \((Ay^*)\) all \( \geq v \Rightarrow R \geq v \).
- Column payoffs \(((x^*)^t A)\) all \( \leq v \Rightarrow v \geq C \).

\( \Rightarrow R \geq C \).

Equilibrium \( \Rightarrow R = C \)!

Strong Duality: There is an equilibrium point! and \( R = C \)!

Doesn't matter who plays first!
Duality.

\[ R = \max_{y} \min_{x} (x^t Ay). \]

\[ C = \min_{x} \max_{y} (x^t Ay). \]

Weak Duality: \[ R \leq C. \]

Proof:

At Equilibrium \((x^*, y^*)\), payoff \(v\):

Row payoffs \((Ay^*)\) all \(\geq v\) \(\Rightarrow R \geq v\).

Column payoffs \((x^* A)\) all \(\leq v\) \(\Rightarrow v \geq C\).

\[ \Rightarrow R \geq C \]

Equilibrium \(\Rightarrow R = C\)!

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**Weak Duality:** \( R \leq C. \)

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At Equilibrium \((x^*, y^*)\), payoff \(v\):
row payoffs \((Ay^*)\) all \(\geq v\) \(\implies R \geq v.\)
Duality.

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\[ C = \min_x \max_y (x^t Ay). \]

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\(\implies R \geq C\)

**Strong Duality:** There is an equilibrium point! Doesn't matter who plays first!
**Duality.**

\[
R = \max_y \min_x (x^t A y).
\]

\[
C = \min_x \max_y (x^t A y).
\]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \(v\):

- row payoffs \((Ay^*)\) all \(\geq v \implies R \geq v.\)
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\(\implies R \geq C\)

Equilibrium \(\implies R = C!\)

**Strong Duality:** There is an equilibrium point!

And \(R = C!\)

Doesn't matter who plays first!
Duality.

\[ R = \max_y \min_x (x^tAy), \]
\[ C = \min_x \max_y (x^tAy). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \) \( \implies \) \( R \geq v \).
column payoffs \(((x^*)^tA)\) all \( \leq v \) \( \implies \) \( v \geq C \).
\( \implies R \geq C \)

Equilibrium \( \implies R = C! \)

**Strong Duality:** There is an equilibrium point!
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

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\[ \implies R \geq C \]

Equilibrium \(\implies R = C!\)

**Strong Duality:** There is an equilibrium point! and \(R = C!\)
Duality.

\[ R = \max_y \min_x (x^tAy) \]
\[ C = \min_x \max_y (x^tAy) \]

**Weak Duality:** \( R \leq C \).

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At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \) \( \implies \) \( R \geq v \).
column payoffs \(((x^*)^tA)\) all \( \leq v \) \( \implies \) \( v \geq C \).
\( \implies \) \( R \geq C \)

Equilibrium \( \implies R = C \)!

**Strong Duality:** There is an equilibrium point! and \( R = C \)!

Doesn’t matter who plays first!
Proof of Equilibrium.

Later. Let’s see some examples.
An “asymptotic” game.

“Catch me.”
An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.
An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.
Given $a, b \in V$. 
An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.
Given $a, b \in V$.
Row (“Catch me”): choose path from $a$ to $b$. 

Column (“Catcher”): choose edge.
Row pays if column chooses edge on path.
An “asymptotic” game.

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Matrix:
row for each path: $p$
An “asymptotic” game.

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row for each path: $p$
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An “asymptotic” game.

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Given $a, b \in V$.
Row ("Catch me"): choose path from $a$ to $b$.
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Matrix:
row for each path: $p$
column for each edge: $e$
Catchme:

Use Blue Path.
Blue with prob. 1/2.
Green with prob. 1/2.
Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:

Caught!
Caught, sometimes.

\(-\frac{1}{2}\)
Catchme:
Use Blue Path.

Catcher:
Catchme:
Use Blue Path.

Catcher:
Caught!
Catchme:
Blue with prob. 1/2.
Green with prob. 1/2.

Catcher:
Catcher:

Caught!

Catchme:

Blue with prob. 1/2.
Green with prob. 1/2.
Catchme:
Blue with prob. 1/3.
Green with prob. 1/6.
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Catcher:
Catchme:
Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:
Caught, sometimes.
With probability 1/2.
Example.


Offense (Best Response):
Catch me: route along shortest path.
(Knows catcher’s distribution.)

Catcher: raise toll on most congested edge.
(Knows catch me’s distribution.)

Defense:
Where should “catcher” play to catch any path?

Minimum cut allows the maximum toll on any edge!

What should “catch me” do to avoid catcher?

minimize maximum load on any edge!

Max-Flow Problem.

Note: exponentially many strategies for “catch me”!

Satish Rao (UC Berkeley)
Example.

Edge solution: $Pr[e_1] = 1/2, Pr[e_2] = 1/2$
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)

Edge solution: \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)

Offense
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)  
Edge solution: \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)

Offense (Best Response.):
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Catch me: route along shortest path.
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Example.

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Where should “catcher” play to catch any path?
Example.

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**Defense:**

Where should “catcher” play to catch any path? a cut.
Example.

Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

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Where should “catcher” play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)
Edge solution: \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)

**Offense (Best Response.):**

Catch me: route along shortest path.
   (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
   (Knows catch me’s distribution.)

**Defense:**

Where should “catcher” play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!

What should “catch me” do to avoid catcher?
Example.

Row solution: \( Pr[p_1] = \frac{1}{2}, \ Pr[p_2] = \frac{1}{3}, \ Pr[p_3] = \frac{1}{6} \).
Edge solution: \( Pr[e_1] = \frac{1}{2}, \ Pr[e_2] = \frac{1}{2} \)

**Offense (Best Response.):**

Catch me: route along shortest path.
   (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
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**Defense:**

Where should “catcher” play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!
What should “catch me” do to avoid catcher?
minimize maximum load on any edge!
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)
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What should “catch me” do to avoid catcher?
minimize maximum load on any edge!
**Max-Flow Problem.**
Example.


Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

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Defense:

Where should “catcher” play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!

What should “catch me” do to avoid catcher?
minimize maximum load on any edge!

**Max-Flow Problem.**

Note: exponentially many strategies for “catch me”!
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
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Row pays if column chooses edge on any path.

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row for each routing: $r$
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Toll/Congestion

Given: $G = (V, E)$.

Given $(s_1, t_1) \ldots (s_k, t_k)$.

Row: choose routing of all paths.

Column: choose edge.

Row pays if column chooses edge on any path.

Matrix:

row for each routing: $r$

column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$
Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll: charge most loaded edge.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1), \ldots, (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll: charge most loaded edge.

**Defense:** Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.
Toll/Congestion

Given: \( G = (V, E) \).
Given \((s_1, t_1) \ldots (s_k, t_k)\).
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: \( r \)
column for each edge: \( e \)

\( A[r, e] \) is congestion on edge \( e \) by routing \( r \)

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll: charge most loaded edge.

**Defense:** Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Again: exponential number of paths for route player.
Summary...

You should now know about

- Games
- Nash Equilibrium
- Pure Strategies
- Zero Sum Two Person Games
- Mixed Strategies
- Checking Equilibrium
- Best Response
- Statement of Duality Theorem
You should now know about Games
Nash Equilibrium
Pure Strategies
Zero Sum Two Person Games
Mixed Strategies.

Checking Equilibrium.

Best Response.

Statement of Duality Theorem.
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...see you Tuesday.