CS270: Lecture 3.

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Path Routing Problem. (Min)

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“Near” optimal solution

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And: Strategic Games

Gradient Descent.

Give differentiable $f(x)$, find minimum.

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Dumber: just move to $x^{(i+1)}$ with smaller $f(x^{(i)}$ in affine subspace.

Routing and Function minimization.

Simple Version of Routing problem.

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Max Congestion Optimal routing, $R*$, has $f(R^*) \leq m 2^{c_{opt}}$.

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$\square$

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What choices do we have?

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Now, we have:
$c = Ax$, minimize $\max_e c(e)$ where $\sum_p x(p) = 1$.

...and smoothing: continued.

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Algorithm: reduce potential!

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Convergence time:

Potential drop: $\geq \varepsilon \sum_{e \in p} 2^{c(e)}$

Move Size: δ.

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Move Size: δ.

Time: Poly$(1/\varepsilon, 1/\delta, n, m)$.

Continuous view: calculus.

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(A) $\nabla(f(x)) = A^t \overrightarrow{2^{c(e)} \ln 2}$?

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(A) $\nabla(f(x)) = A^t \overrightarrow{2^{c(e)} \ln 2}$? or (B) $\nabla(f(x)) = A \overrightarrow{2^{c(e)} \ln 2}$?

(A). Produces a vector of same dimension as x!

$c = Ax$

Continuous view: calculus.

$c = Ax$, minimize $\max_e c(e)$ where $\sum_p x(p) = F$.

$A[e,p]$ - 1 if $e \in p$, 0 otherwise.

c is indexed by e or has dimension m.
x is indexed by p or has dimension total number of s-t paths.

Smooth version: x that minimizes $\sum_e 2^{c(e)}$

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Projection.

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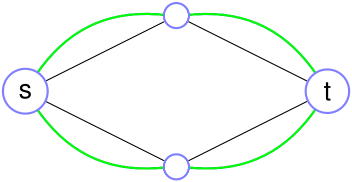
Affine subspace: so can project!

Picture



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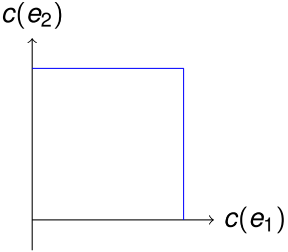
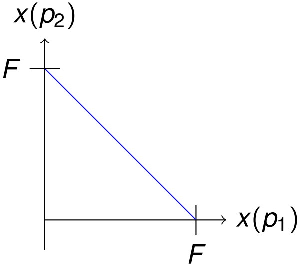
e space isocline.



$\leftarrow A = I \rightarrow$

x space feasibility.



Strategic Games.

N players.

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Each player has strategy set. $\{S_1, \ldots, S_N\}$.

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Player 1: { **D**efect, **C**ooperate }.
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Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

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Nash Equilibrium:

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Defects! Payoff $(.1, .1)$.

Stable now!

Nash Equilibrium:
neither player has incentive to change strategy.

Digression..

What situations?

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Prisoner's dilemma:

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Two prisoners separated by jailors and asked to betray partner.

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Today: simpler version.

Two Person Zero Sum Games

2 players.

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Each player has strategy set:

Two Person Zero Sum Games

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Each player has strategy set:
m strategies for player 1

Two Person Zero Sum Games

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“Player 1 pays *a* to player 2.”

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Payoffs by *m* by *n* matrix: *A*.
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Roshambo: rock,paper, scissors.

	R	P	S
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Any Nash Equilibrium?

(R,R)? no. (R,P)? no. (R,S)? no. ...

Mixed Strategies.

		R	P	S
R		0	1	-1
P		-1	0	1
S		1	-1	0

How do you play?

Mixed Strategies.

		R	P	S
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

How do you play?

Player 1: play each strategy with equal probability.

Mixed Strategies.

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Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Mixed Strategies.

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		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Player 1: play each strategy with equal probability.
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Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Payoffs?

Payoffs: Equilibrium.

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Payoffs? Can't just look it up in matrix!.

Payoffs: Equilibrium.

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Payoffs? Can't just look it up in matrix!.

Average Payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently:

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X]$$

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j)$$

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j) = 0.$$

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

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Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j) = 0.$$

Payoff for other player?

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j) = 0.$$

Payoff for other player? One payoff!

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j) = 0.$$

Payoff for other player? One payoff!
- row minimizes.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j) = 0.$$

Payoff for other player? One payoff!
- row minimizes. column maximizes.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0.$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0.$

No better pure strategy.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0.$

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j)$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0.$

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
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No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3}\times 0+\frac{1}{3}\times 1+\frac{1}{3}\times -1=0.$
Expected payoff of Paper? $\frac{1}{3}\times -1+\frac{1}{3}\times 0+\frac{1}{3}\times 1=0.$
Expected payoff of Scissors? $\frac{1}{3}\times 1+\frac{1}{3}\times -1+\frac{1}{3}\times 0=0.$

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X]=\sum_{(i,j)}(Pr[i]\times Pr[j])X(i,j)=\sum_i Pr[i](\sum_j Pr[j]\times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change!

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)}(Pr[i] \times Pr[j])X(i,j) = \sum_i Pr[i](\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0.$

No better pure strategy. $\implies$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium!

Another example plus notation.

Rock, Paper, Scissors, prEempt.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium?

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation:

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
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Both only play optimal strategies!

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Why play more than one?

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Both only play optimal strategies! Complementary slackness.
Why play more than one? Limit opponent payoff!

Two person zero sum games.

$m \times n$ payoff matrix A.

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Payoff for strategy pair (x, y):

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Payoff for strategy pair (x, y):

$$p(x, y) = x^t A y$$

That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j}$$

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$$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

Two person zero sum games.

$m \times n$ payoff matrix A.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair (x, y):

$$p(x, y) = x^t A y$$

That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j} = \sum_i x_i \left(\sum_j a_{i,j} y_j \right) = \sum_i \sum_j x_i a_{i,j} y_j = \sum_j \left(\sum_i x_i a_{i,j} \right) y_j.$$

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(No better column strategy, no better row strategy.)

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$\min_i A^{(i)} \cdot y = (x^*)^t A y^*$. [1]

No column is better:
$\max_j (A^t)^{(j)} \cdot x = (x^*)^t A y^*$.

[1] $A^{(i)}$ is ith row.

Best Response

Column goes first:

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Find y, where best row is not too low..

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Row goes first:
Find x, where best column is not high.

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Example: Roshambo. Value of R?

Row goes first:
Find x, where best column is not high.

$$C = \min_x \max_y (x^t A y).$$

Best Response

Column goes first:
Find y, where best row is not too low..

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Row goes first:
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Example: Roshambo. Value of C?

Duality.

$$R = \max_y \min_x (x^t A y).$$

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Weak Duality: $R \le C$.
Proof: Better to go second. □

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row payoffs (Ay^*) all $\geq v \implies R \geq v$.

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column payoffs $((x^*)^t A)$ all $\leq v$

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$\implies R \ge C$

Equilibrium $\implies R = C$!

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Strong Duality: There is an equilibrium point!

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$\implies R \geq C$

Equilibrium $\implies R = C$!

Strong Duality: There is an equilibrium point! and $R = C$!

Doesn't matter who plays first!

Proof of Equilibrium.

Later. Let's see some examples.

An “asymptotic” game.

“Catch me.”

An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.

An "asymptotic" game.

"Catch me."

Given: $G = (V, E)$.
Given $a, b \in V$.

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"Catch me."

Given: $G = (V, E)$.
Given $a, b \in V$.
Row ("Catch me"): choose path from a to b.

An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.
Given $a, b \in V$.
Row (“Catch me”): choose path from a to b.
Column(“Catcher”): choose edge.

An "asymptotic" game.

"Catch me."

Given: $G = (V, E)$.
Given $a, b \in V$.
Row ("Catch me"): choose path from a to b.
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Row pays if column chooses edge on path.

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Matrix:
row for each path: p

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An "asymptotic" game.

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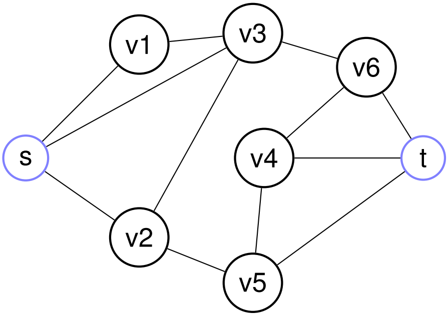
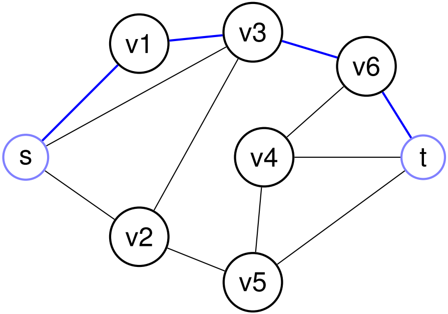
Row pays if column chooses edge on path.

Matrix:
row for each path: p
column for each edge: e
$A[p, e] = 1$ if $e \in p$.



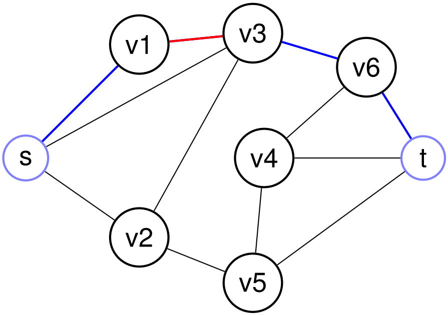
Catchme:

Catcher:



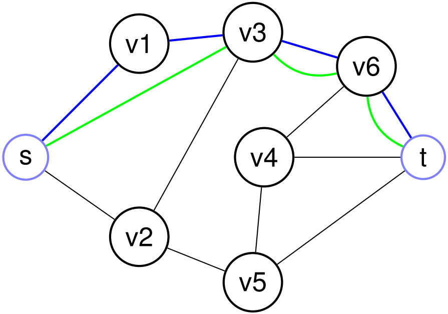
Catchme:
Use Blue Path.

Catcher:



Catchme:
Use Blue Path.

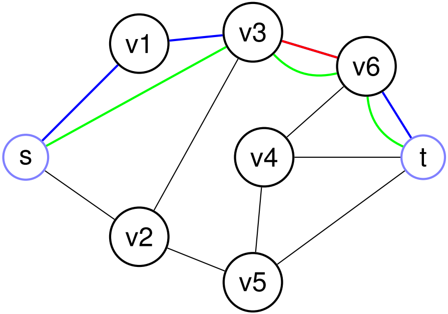
Catcher:
Caught!



Catchme:

Blue with prob. 1/2.
Green with prob. 1/2.

Catcher:



Catchme:

Blue with prob. 1/2.
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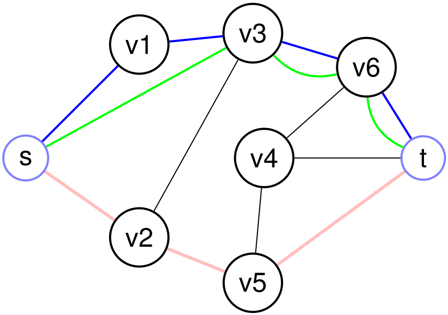
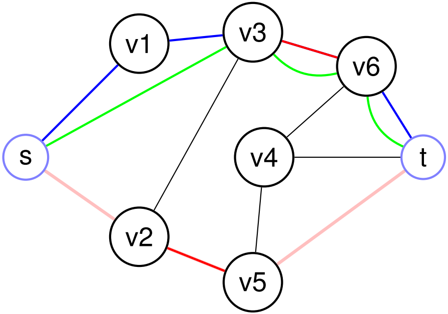
Catcher:
Caught!



Catchme:

Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:



Catchme:

Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:
Caught, sometimes.
With probability $1/2$.

Example.

Row solution: $Pr[p_1] = 1/2$, $Pr[p_2] = 1/3$, $Pr[p_3] = 1/6$.

Example.

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Offense

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Offense (Best Response.):

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Catch me: route along shortest path.

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Offense (Best Response.):

Catch me: route along shortest path.
(Knows catcher's distribution.)

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Row solution: $Pr[p_1] = 1/2$, $Pr[p_2] = 1/3$, $Pr[p_3] = 1/6$.
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Offense (Best Response.):

Catch me: route along shortest path.
(Knows catcher's distribution.)
Catcher: raise toll on most congested edge.

Example.

Row solution: $Pr[p_1] = 1/2$, $Pr[p_2] = 1/3$, $Pr[p_3] = 1/6$.
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Catch me: route along shortest path.
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Catcher: raise toll on most congested edge.
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Defense:

Where should "catcher" play to catch any path?

Example.

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Defense:

Where should "catcher" play to catch any path? a cut.

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Catcher: raise toll on most congested edge.
(Knows catch me's distribution.)

Defense:

Where should "catcher" play to catch any path? a cut.
Minimum cut allows the maximum toll on any edge!

Example.

Row solution: $Pr[p_1] = 1/2$, $Pr[p_2] = 1/3$, $Pr[p_3] = 1/6$.
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Defense:

Where should "catcher" play to catch any path? a cut.
Minimum cut allows the maximum toll on any edge!

What should "catch me" do to avoid catcher?

Example.

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minimize maximum load on any edge!

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Max-Flow Problem.

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Minimum cut allows the maximum toll on any edge!

What should "catch me" do to avoid catcher?
minimize maximum load on any edge!
Max-Flow Problem.

Note: exponentially many strategies for "catch me"!

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

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$A[r, e]$ is congestion on edge e by routing r

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Offense: (Best Response.)

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$A[r,e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.

Toll/Congestion

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$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
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row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
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Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Toll/Congestion

Given: $G = (V, E)$.
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Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Again: exponential number of paths for route player.

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Games

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Nash Equilibrium

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Finding Equilibrium.

...see you Tuesday.