CS270: Lecture 3.

Last Time:
- Path Routing Problem. (Min)
- Toll Problem. (Max)

Toll $\leq$ Path.

Algs: Exp. Weights for Tolls/Shortest Paths for Path.

"Near" optimal solution s !

Today: continuous view.

And: Strategic Games

Gradient Descent.

Give differentiable $f(x)$, find minimum.

Alg:
While "not good enough":
$x^{i+1} = x^i - \varepsilon_i \nabla f(x^i)$.

$\nabla(f(x^i)) = 0 \implies$ Optimal.

Constrained: project gradient into affine space.

Projected$(\nabla(f(x^i)) = 0 \implies$ Optimal.

Dumber: just move to $x^{(i+1)}$ with smaller $f(x^{(i)}$ in affine subspace.

Routing and Function minimization.

Simple Version of Routing problem.
Route X units of flow between s and t.
Minimize congestion.

minimize $\max_e c(e)$. Not smooth.

Smoothing functions: minimize $\max_e c(e)$.
$f(R) = \sum_e 2^{c(e)}$
$f'(R) = \sum_e c(e) 2^{c(e)}$

Good smoothing?
Thm: Routing R that minimizes $f(R)$ has $\max_e c(e) = c(R) \leq c_{opt} + \log m$.

Proof:
Max Congestion Optimal routing, $R*$, has $f(R^*) \leq m2^{c_{opt}}$.
Why? m edges each with congestion at most c_{opt}.
This routing has $f(R) \geq 2^{c(R)}$.

$\rightarrow m2^{c_{opt}} \geq f(R) \geq 2^{c(R)}$.
$\rightarrow c_{opt} + \log m \geq c(R)$. □

Optimization Setup: continued.

R "routes" a F units of flow for one pair (s,t).

$\nabla f(R) = c'(e) 2^{c(e)} \log 2$.

With respect to what?
What are the variables?
What choices do we have?

As optimization: continued

R "routes" a unit flow for one pair (s,t).

"Decision Variable".
For an $s-t$ path p, $x(p)$ flow along p.

Exponential number! Uh oh?

Constraint: sum of $x(p)$ is 1.

What is $c(e)$ in terms of $x(p)$?

$A[e,p] = 1$ if $e \in p$ and 0 otherwise.

Now, we have:
$c = Ax$, minimize $\max_e c(e)$ where $\sum_p x(p) = 1$.

...and smoothing: continued.

Now, we have:
$c = Ax$, minimize $\max_e c(e)$ where $\sum_p x(p) = F$.

Smooth version: minimize $\sum_e 2^{c(e)}$.

Minimum gives solution within additive $\log m$ of optimal.
Better?: F to $2F \implies$ error divides by two.
F to $F/\delta \implies$ additive error is $\delta \log m$.

Algorithm: reduce potential! $\sum_e 2^{c(e)}$.

Best possible: a factor of two off.
Oscillates if move when length of path not smaller by factor of 2.

$$\sum_e 2^{c(e)} \rightarrow \sum_e (1+\varepsilon)^{c(e)}.$$

Approximate Equilibrium: $(1+2\varepsilon)C_{opt} + \delta \log n/\varepsilon$.

Convergence time:
Potential drop: $\geq \varepsilon \sum_{e \in p} 2^{c(e)}$
Move Size: δ.

Time: Poly$(1/\varepsilon, 1/\delta, n, m)$.

Continuous view: calculus.

$c = Ax$, minimize $\max_e c(e)$ where $\sum_p x(p) = F$.

$A[e,p]$ - 1 if $e \in p$, 0 otherwise.

c is indexed by e or has dimension m.
x is indexed by p or has dimension total number of s-t paths.

Smooth version: x that minimizes $\sum_e 2^{c(e)}$

Variables are vector x, indexed by path p.

So what is gradient?

(A) $\nabla(f(x)) = A^t\, \overrightarrow{2^{c(e)} \ln 2}$? or (B) $\nabla(f(x)) = A\, \overrightarrow{2^{c(e)} \ln 2}$?

(A). Produces a vector of same dimension as x!

$$c = Ax \implies \frac{\partial c(e)}{\partial(x(p))} = A[e,p] \implies \frac{\partial \sum_e 2^{c(e)}}{\partial(x(p))} \propto \sum_e 2^{c(e)} \frac{\partial c(e)}{\partial(x(p))} = (A^t)^{(p)} \cdot \overrightarrow{2^{c(e)}}$$

$$\implies \nabla_x(f(R)) \propto A^t \overrightarrow{2^{c(e)}}.$$

Projection.

$c = Ax$, minimize $\max_e c(e)$ where $\sum_p x(p) = F$.

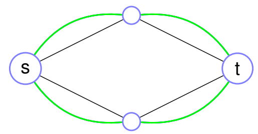
Smooth version: x that minimizes $\sum_e 2^{c(e)}$

$\nabla_x(f(R)) \propto A^t \overrightarrow{2^{c(e)}}$.

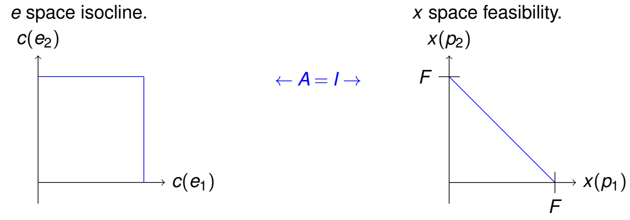
We also have: $\sum_p x(p) = F$

Affine subspace: so can project!

Picture



$c = Ax$



Strategic Games.

N players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \Re^N$).

Example:

2 players

Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(.1,1)

What is the best thing for the players to do?

Both cooperate. Payoff $(3,3)$.

If player 1 wants to do better, what does she do?

Defects! Payoff $(5,0)$

What does player 2 do now?

Defects! Payoff $(.1,.1)$.

Stable now!

Nash Equilibrium:
neither player has incentive to change strategy.

Digression..

What situations?

Prisoner's dilemma:
Two prisoners separated by jailors and asked to betray partner.

Basis of the free market.
Companies compete, don't cooperate.
No Monopoly:
E.G., OPEC, Airlines, .
Should defect.
Why don't they?
Free market economics ...not so much?
More sophisticated models ,e.g, iterated dominance, coalitions, complexity..
Lots of interesting Game Theory!

Today: simpler version.

Two Person Zero Sum Games

2 players.

Each player has strategy set:
m strategies for player 1
n strategies for player 2

Payoff function: $u(i,j) = (-a, a)$ (or just a).
"Player 1 pays a to player 2."

Zero Sum: Payoff for any pair of strategies sums to 0.

Payoffs by m by n matrix: A.
Row player minimizes, column player maximizes.

Roshambo: rock,paper, scissors.

	R	P	S
R	0	1	-1
P	-1	0	1
S	1	-1	0

Any Nash Equilibrium?

(R,R)? no. (R,P)? no. (R,S)? no. ...

Mixed Strategies.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

Each player chooses independently: $Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = \frac{1}{9} \sum_{(i,j)} X(i,j) = 0.$$

Payoff for other player? One payoff!
- row minimizes. column maximizes.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0$.

No better pure strategy. $\Longrightarrow$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium!

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Playing the boss...

Row has extra strategy:Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note: column knows row cheats.
Why play?
Row is column's advisor.
... boss.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equilibrium: Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Payoff? Remember: weighted average of pure strategies.

Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}$
Strategy 2: $\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}$
Strategy 3: $\frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 = -\frac{1}{6}$
Strategy 4: $\frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times -1 = -\frac{1}{6}$

Payoff is $0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6}) = -\frac{1}{6}$

Column player: every column payoff is $-\frac{1}{6}$.

Both only play optimal strategies! Complementary slackness.
Why play more than one? Limit opponent payoff!

Two person zero sum games.

$m \times n$ payoff matrix A.

Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair (x, y):

$$p(x, y) = x^t A y$$

That is,

$$\sum_{i,j} (x_i y_j) \cdot a_{i,j} = \sum_i x_i \left(\sum_j a_{i,j} y_j \right) = \sum_i \sum_j x_i a_{i,j} y_j = \sum_j \left(\sum_i x_i a_{i,j} \right) y_j.$$

Recall row minimizes, column maximizes.

Equilibrium pair: (x^*, y^*)?

$$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

Equilibrium.

Equilibrium pair: (x^*, y^*)?

$$p(x, y) = (x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

No row is better:
$\min_i A^{(i)} \cdot y = (x^*)^t A y^*$. [1]

No column is better:
$\max_j (A^t)^{(j)} \cdot x = (x^*)^t A y^*$.

[1] $A^{(i)}$ is ith row.

Best Response

Column goes first:
Find y, where best row is not too low..

$$R = \max_y \min_x (x^t A y).$$

Note: x can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of R?

Row goes first:
Find x, where best column is not high.

$$C = \min_x \max_y (x^t A y).$$

Agin: y of form $(0, 0, \ldots, 1, \ldots 0)$. From Texas.

Example: Roshambo. Value of C?

Duality.

$$R = \max_y \min_x (x^t A y).$$
$$C = \min_x \max_y (x^t A y).$$

Weak Duality: $R \leq C$.
Proof: Better to go second. □

At Equilibrium (x^*, y^*), payoff v:
row payoffs (Ay^*) all $\geq v \implies R \geq v$.
column payoffs $((x^*)^t A)$ all $\leq v \implies v \geq C$.
$\implies R \geq C$

Equilibrium $\implies R = C$!

Strong Duality: There is an equilibrium point! and $R = C$!

Doesn't matter who plays first!

Proof of Equilibrium.

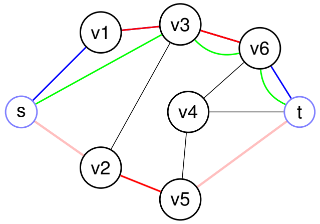
Later. Let's see some examples.

An "asymptotic" game.

"Catch me."

Given: $G = (V, E)$.
Given $a, b \in V$.
Row ("Catch me"): choose path from a to b.
Column("Catcher"): choose edge.
Row pays if column chooses edge on path.

Matrix:
row for each path: p
column for each edge: e
$A[p, e] = 1$ if $e \in p$.



Catchme:
Use Blue Path.
Blue with prob. 1/2.
Blue with prob. 1/3.
Green with prob. 1/2.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:
Caught!
Caught, sometimes.
With probability $1/2$.

Example.

Row solution: $Pr[p_1] = 1/2$, $Pr[p_2] = 1/3$, $Pr[p_3] = 1/6$.
Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

Offense (Best Response.):

Catch me: route along shortest path.
(Knows catcher's distribution.)
Catcher: raise toll on most congested edge.
(Knows catch me's distribution.)

Defense:

Where should "catcher" play to catch any path? a cut.
Minimum cut allows the maximum toll on any edge!

What should "catch me" do to avoid catcher?
minimize maximum load on any edge!
Max-Flow Problem.

Note: exponentially many strategies for "catch me"!

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Again: exponential number of paths for route player.

Summary...

You should now know about

Games
Nash Equilibrium
Pure Strategies
Zero Sum Two Person Games
Mixed Strategies.
Checking Equilibrium.
Best Response.
Statement of Duality Theorem.

Finding Equilibrium.

...see you Tuesday.