Restricted Isometry Property (RIP) matrices.

**Definition:** A matrix $A$ is RIP for $\delta_k$ if any $k$-sparse vector $x$ satisfies
\[
(1 - \delta_k)\|x\|_2 \leq \|Ax\|_2 \leq (1 + \delta_k)\|x\|_2.
\]

**Theorem (Candes-Tao):** For any matrix RIP matrix $A$ with $\delta_2k + \delta_3k < 1$, for $Ax = b$ with a $k$-sparse solution, then the solution to $\min\|y\|_1, Ay = b$, has $y = x$.

Almost Euclidean Nullspace.

**Theorem:** For a random $\pm 1, d \times n$ matrix $A$, and for any $x \in \ker(A)$ some $d = \Omega(k \log n/k)$ rows, has for any $T \subset [n]$ that
\[
\|x\|_2 < \frac{\sqrt{d\log n}}{2k}\|x\|_1.
\]

Intuition: “Mass in $x$ is spread out over $k$ entries.”

The nullspace of $A$, is almost euclidean.

Typical vectors are spread out: every vector is kind of spread out.

The $\ell_1$ ball is closer to scaling of $\ell_2$ ball for vectors in the null-space.

Idea: Consider random $r \times n$ matrix $A$ over $GF(2)$.

For a vector $x$ in $GF(2)$, $A_x = 0$, with probability $(1/2)^r$ if $r$ rows.

There are $< X = 2^{(\log n)/k}$ vectors $x$ with fewer than $k$ zeros.

If $r > \log(2^{(\log n)/k}) = O(k \log 2)$, plus union bound.

$Ax = 0$ for all vectors that are $k$-sparse.

That is, random $A$ has no sparse vectors in null-space.

Note: Parity check matrix of linear code!

Compressed Sensing.

Find $x$ with small number of non-zeros using linear measurements.

$Ax = b$.

Application: MRL

Find $x$ with $k$-sparse $x$, i.e., $\supp(x) \leq k$.

$\ell_0$-minimization.

Extremely “non-convex”.

Find solution to $\min\|w\|_1, Ax = b$.

Linear Program! Exercise.

Fun with $\ell_1$ and $\ell_2$

$\|x\|_1 \leq \sqrt{n}\|x\|_2$.

$\|x\|_1 = x \cdot \text{sgn}(x) \leq \|x\|_2\|\text{sgn}(x)\|_2 \leq \sqrt{\|x\|_2}$

$\|x\|_1 = x \cdot \text{sgn}(x) \leq \|x\|_2\|\text{sgn}(x)\|_2 \leq \sqrt{\|\supp(x)\|_2}\|x\|_2$

$\supp(x)$ is non-zero indices of $x$.

If concentrated mass, $\|x\|_1 = \|x\|_2$.

$x = (1, 0, 0, \ldots, 0)$.

If spreadout, $\sqrt{n}\|x\|_2 \leq \|x\|_1$.

$x = (1, 1, 1, \ldots, 1)$.

If kind of spread out, $\|x\|_2 \leq \frac{1}{\sqrt{n}}\|x\|_1$.

$x$ has $k$ 1’s.

Fixing $\|v\|_2$, sparse vectors have small $\|v\|_1$ norm, dense ones have big $\|v\|_1$ norm.

Small projection onto small set of coordinates.

Consider $A$ with property, $x \in \ker(A)$, has $\|x\|_1 < \frac{1}{16\sqrt{k}}\|x\|_1$.

**Lemma:** For $v \in \ker(A)$, $T \subset [n], |T| < k$,

$\|v_T\| \leq \frac{\|v\|_1}{2^{(\log n)/k}}$.

Intuition:

For any $v \in \ker(A)$, the amount of mass in any small, $k$, set of coordinates is small, $\frac{1}{2^{(\log n)/k}}$.

Mass is spread out over more than $k$ coordinates.
Proof of $\|w - x\| \leq 4\sigma(x)$.

Again: $\sigma_k(x) = \min_{\text{supp}(z) \leq k} |x - z|_1$.

Lemma: For $v \in \ker(A), \ T \subset [n], |T| \leq k$.

Theorem: $T$ be $k$ largest in magnitude coordinates of $x$.

Proof of Theorem: $T$ be $k$ largest in magnitude coordinates of $x$.

"Few" vectors with most of mass in small set of coordinates.

Union bound over those.

Credits


Possible Topics.

TODO: Long tailed distributions.

Interior Point Algorithms.

Matrix Concentration/Matrix Experts/Semidefinite Programs.

Coding Theory: Low Density Parity Check Codes or Expander codes.

Auctions. Mechanism Design.