Strategic Games.

$N$ players.

Each player has strategy set: 

$\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \mathbb{R}^N$).

Example: 2 players

Player 1: $\{\text{Defect, Cooperate}\}$.

Player 2: $\{\text{Defect, Cooperate}\}$.

Payoff:

$\begin{align*}
\text{CD} & : (3, 3) \\
\text{DC} & : (0, 5) \\
\text{DD} & : (5, 0) \\
\text{CC} & : (1, 1)
\end{align*}$
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What is the best thing for the players to do?
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Both cooperate. Payoff (3,3).

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If player 1 wants to do better, what does she do?

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Stable now!

Nash Equilibrium: neither player has incentive to change strategy.
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Proving Nash.

$n$ players.
Proving Nash.

\[ n \text{ players.} \]

Player \( i \) has strategy set \( \{1, \ldots, m_i\} \).
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Player $i$ has strategy set $\{1, \ldots, m_i\}$.

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Theorem: There is a Nash Equilibrium.
Brouwer Fixed Point Theorem.

**Theorem:** Every continuous function from a closed compact convex (c.c.c.) set to itself has a fixed point.

![Diagram](image_url)

What is the closed convex set here? The unit square? Or the unit interval?
Brouwer Fixed Point Theorem.

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What is the closed convex set here?
**Brouwer Fixed Point Theorem.**

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Brouwer implies Nash.

The set of mixed strategies $x$ is closed convex set.
Brouwer implies Nash.

The set of mixed strategies \( x \) is closed convex set. That is, \( x = (x_1, \ldots, x_n) \) where \( |x_i|_1 = 1 \).
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Unique minimum as quadratic. $z_i$ is continuous in $x$. Mixed strategy utilities is polynomial of entries of $x$ with coefficients being payoffs in game matrix. $\phi(\cdot)$ is continuous on the closed convex set.

Brouwer: Has a fixed point: $\phi(\hat{z}) = \hat{z}$. 
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Consider \( \hat{y}_i = \hat{z}_i + \alpha(y_i - z_i). \)
\[ u_i(\hat{z}_{-i}; \hat{y}_i) + \| \hat{z}_i - y_i \|^2? \]
\[ u_i(\hat{z}) + \alpha(u_i(\hat{z}) + \delta - u_i(\hat{z})) - \alpha^2 \| \hat{z}_i - y_i \|^2 \]
\[ = u_i(\hat{z}) + \alpha \delta - \alpha^2 \| y_i - \hat{z}_i \|^2 > u_i(\hat{z}). \]

The last inequality true when \( \alpha < \frac{\delta}{\| y_i - z_i \|^2}. \)
Fixed Point is Nash.

\[ \phi(x_1, \ldots, x_n) = (z_1, \ldots, z_n) \] where

\[ z_i = \arg \max_{z_i'} \left[ u_i(x_{-i}; z_i') + \| z_i - x_i \|^2 \right]. \]

Fixed point: \( \phi(\hat{z}) = \hat{z} \)

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Thus, \( \hat{z} \) not a fixed point!
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Fixed Point is Nash.

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\[ z_i = \arg \max_{z'_i} \left[ u_i(x_{-i}; z'_i) + \| z_i - x_i \|^2_2 \right]. \]

Fixed point: \( \phi(\hat{z}) = \hat{z} \)

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Thus, \( \hat{z} \) not a fixed point!

Thus, fixed point is Nash. \( \square \)
Sperner’s Lemma

For any $n+1$-dimensional simplex which is subdivided into smaller simplices.
Sperner’s Lemma

For any $n+1$-dimensional simplex which is subdivided into smaller simplices.

All vertices are colored $\{1,\ldots, n+1\}$. 
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The coloring is proper if the extremal vertices are differently colored.
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The coloring is proper if the extremal vertices are differently colored.

Each face only contains the colors of the incident corners.
Sperner’s Lemma

For any \( n + 1 \)-dimensional simplex which is subdivided into smaller simplices.

All vertices are colored \( \{1, \ldots, n + 1\} \).

The coloring is proper if the extremal vertices are differently colored. Each face only contains the colors of the incident corners.

**Lemma:** There exist a simplex that has all the colors.
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**Lemma:** There exist a simplex that has all the colors.

![Diagram of a triangle with colored vertices]
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The coloring is proper if the extremal vertices are differently colored.

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**Lemma:** There exist a simplex that has all the colors.

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Where is multicolored?
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Where is multicolored?
Where is multicolored? And now?
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For any $n + 1$-dimensional simplex which is subdivided into smaller simplices.

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Where is multicolored?

Where is multicolored? And now?

By induction!
Proof of Sperner’s.

One dimension:
Proof of Sperner’s.

One dimension: Subdivision of [0, 1].
Proof of Sperner’s.

One dimension: Subdivision of \([0, 1]\).
Endpoints colored differently.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
Endpoints colored differently.
Odd number of multicolored edges.
Proof of Sperner’s.

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One dimension: Subdivision of $[0, 1]$.
Endpoints colored differently.
Odd number of multicolored edges.

Two dimensions.
Proof of Sperner’s.

One dimension: Subdivision of $[0,1]$.
   Endpoints colored differently.
   Odd number of multicolored edges.

Two dimensions.
   Consider $(1,2)$ edges.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.

Endpoints colored differently.
Odd number of multicolored edges.

Two dimensions.
Consider $(1, 2)$ edges.
Separates two regions.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
   Endpoints colored differently.
   Odd number of multicolored edges.

Two dimensions.
Consider $(1, 2)$ edges.
   Separates two regions.
   Dual edge connects regions with 1 on right.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.

Endpoints colored differently.
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Two dimensions.
Consider $(1, 2)$ edges.
Separates two regions.
Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
   Endpoints colored differently.
   Odd number of multicolored edges.

Two dimensions.
   Consider $(1, 2)$ edges.
   Separates two regions.
   Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
   one more $(1, 2)$ than $(2, 1)$. 
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
  Endpoints colored differently.
  Odd number of multicolored edges.

Two dimensions.
  Consider $(1, 2)$ edges.
  Separates two regions.
    Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
  one more $(1, 2)$ than $(2, 1)$.
There exist a region with excess in-degree.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
   Endpoints colored differently.
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Two dimensions.
   Consider $(1, 2)$ edges.
   Separates two regions.
   Dual edge connects regions with 1 on right.

Exterior region has excess out-degree:
   one more $(1, 2)$ than $(2, 1)$.

There exist a region with excess in-degree.
   $(1, 2, 1)$ triangle has in-degree=out-degree.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
   Endpoints colored differently.
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Two dimensions.
   Consider $(1, 2)$ edges.
   Separates two regions.
   Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
   one more $(1, 2)$ than $(2, 1)$.
There exist a region with excess in-degree.
   $(1, 2, 1)$ triangle has in-degree= outgoing.
   $(2, 1, 2)$ triangle has in-degree= outgoing.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
   Endpoints colored differently.
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Two dimensions.
   Consider $(1, 2)$ edges.
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Exterior region has excess out-degree:
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There exist a region with excess in-degree.
   $(1, 2, 1)$ triangle has in-degree=out-degree.
   $(2, 1, 2)$ triangle has in-degree=out-degree.

Must be $(1, 2, 3)$ triangle.
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.

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Odd number of multicolored edges.

Two dimensions.
Consider $(1, 2)$ edges.
Separates two regions.
Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
one more $(1, 2)$ than $(2, 1)$.
There exist a region with excess in-degree.
$(1, 2, 1)$ triangle has in-degree=out-degree.
$(2, 1, 2)$ triangle has in-degree=out-degree.
Must be $(1, 2, 3)$ triangle.
Must be odd number!
Proof of Sperner’s.

One dimension: Subdivision of $[0, 1]$.
Endpoints colored differently.
Odd number of multicolored edges.

Two dimensions.
Consider $(1, 2)$ edges.
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Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
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There exist a region with excess in-degree.
$(1, 2, 1)$ triangle has in-degree=out-degree.
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Must be odd number!
Proof of Sperner’s.

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   Consider $(1,2)$ edges.
   Separates two regions.
   Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
   one more $(1,2)$ than $(2,1)$.
There exist a region with excess in-degree.

$(1,2,1)$ triangle has in-degree=out-degree.
$(2,1,2)$ triangle has in-degree=out-degree.

Must be $(1,2,3)$ triangle.
Must be odd number!
$n+1$-dimensional Sperner.

$R$: counts “rainbow” cells; has all $n+1$ colors.
$n+1$-dimensional Sperner.

$R$: counts “rainbow” cells; has all $n+1$ colors.
$Q$: counts “almost rainbow” cells;
$n+1$-dimensional Sperner.

$R$: counts “rainbow” cells; has all $n+1$ colors.

$Q$: counts “almost rainbow” cells; has $\{1, \ldots, n\}$.
$n+1$-dimensional Sperner.

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Note: exactly one color in $\{1, \ldots, n\}$ used twice.
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Rainbow face: $n-1$-dimensional, vertices colored with $\{1, \ldots, n\}$. 

Number of Face-Rainbow Cell Adjacencies:

$$R + 2Q = X + 2Y$$

Rainbow faces on one face of big simplex.

Induction $\Rightarrow$ Odd number of rainbow faces.
$\rightarrow$ $X$ is odd $\rightarrow$ $X + 2Y$ is odd

$R$ is odd.
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Rainbow face: $n-1$-dimensional, vertices colored with $\{1, \ldots, n\}$.
   $X$: number of boundary rainbow faces.
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   Note: exactly one color in $\{1, \ldots, n\}$ used twice.

Rainbow face: $n-1$-dimensional, vertices colored with $\{1, \ldots, n\}$.
   $X$: number of boundary rainbow faces.
   $Y$: number of internal rainbow faces.
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Rainbow face: $n-1$-dimensional, vertices colored with $\{1,\ldots,n\}$.
   $X$: number of boundary rainbow faces.
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Number of Face-Rainbow Cell Adjacencies: $R + 2Q = X + 2Y$

Rainbow faces on one face of big simplex.
   Induction $\implies$ Odd number of rainbow faces.
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Number of Face-Rainbow Cell Adjacencies: $R + 2Q = X + 2Y$

Rainbow faces on one face of big simplex.

Induction $\implies$ Odd number of rainbow faces.

$\rightarrow$ $X$ is odd
$n+1$-dimensional Sperner.

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Rainbow face: $n-1$-dimensional, vertices colored with $\{1,\ldots,n\}$.

$X$: number of boundary rainbow faces.

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Number of Face-Rainbow Cell Adjacencies: $R + 2Q = X + 2Y$

Rainbow faces on one face of big simplex.

Induction $\implies$ Odd number of rainbow faces.

$\implies X$ is odd $\implies X + 2Y$ is odd
$n+1$-dimensional Sperner.

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Rainbow face: $n-1$-dimensional, vertices colored with $\{1, \ldots, n\}$.
   $X$: number of boundary rainbow faces.
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Number of Face-Rainbow Cell Adjacencies: $R + 2Q = X + 2Y$

Rainbow faces on one face of big simplex.
   Induction $\implies$ Odd number of rainbow faces.
   $\rightarrow X$ is odd $\rightarrow X + 2Y$ is odd $R + 2Q$ is odd.
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Rainbow face: $n - 1$-dimensional, vertices colored with $\{1, \ldots, n\}$.
   $X$: number of boundary rainbow faces.
   $Y$: number of internal rainbow faces.

Number of Face-Rainbow Cell Adjacencies: $R + 2Q = X + 2Y$

Rainbow faces on one face of big simplex.
   Induction $\implies$ Odd number of rainbow faces.
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Number of Face-Rainbow Cell Adjacencies: $R + 2Q = X + 2Y$

Rainbow faces on one face of big simplex.

Induction $\implies$ Odd number of rainbow faces.

$\rightarrow$ $X$ is odd $\rightarrow$ $X + 2Y$ is odd $R + 2Q$ is odd.

$R$ is odd.
Sperner to Brouwer

Consider simplex: S.
Sperner to Brouwer

Consider simplex: $S$.
Closed compact sets can be mapped to this.
Let $f(x) : S \rightarrow S$. 

Infinite sequence of subdivisions: $S_1, S_2, \ldots$ 
$x_j$ is subdivision of $S$.
Size of cell $\rightarrow 0$ as $j \rightarrow \infty$.

A coloring of $S_j$.
Recall $\sum_i x_i = 1$ in simplex.
Big simplex vertices $e_j = (0, 0, \ldots, 1, \ldots, 0)$ get $j$.
For a vertex at $x$, Assign smallest $i$ with $f(x)_i < x_i$.
Yes.
Valid?
Simplex face is at $x = 0$ for opposite $j$.
Thus $f(x)_j$ cannot be smaller and is not colored $j$.
Rainbow cell, in $S_j$ with vertices $x_j, 1, \ldots, x_j, n + 1$. 
Consider simplex $S$.
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Sperner to Brouwer

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$S_j$ is subdivision of $S_{j-1}$.
Sperner to Brouwer

Consider simplex: $S$.
Closed compact sets can be mapped to this.
Let $f(x) : S \to S$.

Infinite sequence of subdivisions: $S_1, S_2, ...$

$S_j$ is subdivision of $S_{j-1}$. Size of cell $\to 0$ as $j \to \infty$. 
Sperner to Brouwer

Consider simplex $S$.
Closed compact sets can be mapped to this.
Let $f(x) : S \to S$.

Infinite sequence of subdivisions: $\mathcal{S}_1, \mathcal{S}_2, \ldots$

$\mathcal{S}_j$ is subdivision of $\mathcal{S}_{j-1}$. Size of cell $\to 0$ as $j \to \infty$.

A coloring of $\mathcal{S}_j$. 

Consider simplex $S$. Closed compact sets can be mapped to this. Let $f(x) : S \to S$.

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Big simplex vertices $e_j = (0, 0, \ldots, 1, \ldots, 0)$ get $j$. 
Sperner to Brouwer

Consider simplex: S.
Closed compact sets can be mapped to this.
Let \( f(x) : S \rightarrow S \).

Infinite sequence of subdivisions: \( S_1, S_2, \ldots \)

\( S_j \) is subdivision of \( S_{j-1} \). Size of cell \( \rightarrow 0 \) as \( j \rightarrow \infty \).

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Sperner to Brouwer

Consider simplex $S$. Closed compact sets can be mapped to this. Let $f(x) : S \rightarrow S$.

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Assign smallest $i$ with $f(x)_i < x_i$. 
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$\mathcal{S}_j$ is subdivision of $\mathcal{S}_{j-1}$. Size of cell $\to 0$ as $j \to \infty$.

A coloring of $\mathcal{S}_j$. Recall $\sum_i x_i = 1$ in simplex.
Big simplex vertices $e_j = (0,0,\ldots,1,\ldots,0)$ get $j$.

For a vertex at $x$.
Assign smallest $i$ with $f(x)_i < x_i$.
Exists?
Sperner to Brouwer

Consider simplex: $S$.
Closed compact sets can be mapped to this.
Let $f(x) : S \to S$.

Infinite sequence of subdivisions: $\mathcal{I}_1, \mathcal{I}_2, \ldots$

$\mathcal{I}_j$ is subdivision of $\mathcal{I}_{j-1}$. Size of cell $\to 0$ as $j \to \infty$.

A coloring of $\mathcal{I}_j$. Recall $\sum_i x_i = 1$ in simplex.
Big simplex vertices $e_j = (0, 0, \ldots, 1, \ldots, 0)$ get $j$.

For a vertex at $x$.
Assign smallest $i$ with $f(x)_i < x_i$.
Exists? Yes.
Sperner to Brouwer

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Let $f(x): S \to S$.

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For a vertex at $x$.

Assign smallest $i$ with $f(x)_i < x_i$.

Exists? Yes. $\sum_i f(x)_i = \sum_i x_i$. 
Consider simplex: $S$.
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Let $f(x): S \rightarrow S$.

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Thus $f(x)_j$ cannot be smaller and is not colored $j$. 
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Rainbow cell, in $S_j$ with vertices $x_j^1, \ldots, x_j^{n+1}$.
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Rainbow Cells to Brower.

Rainbow cell, in $S_j$ with vertices $x^{j,1}, \ldots, x^{j,n+1}$. 
Rainbow Cells to Brower.

Rainbow cell, in $S_j$ with vertices $x_j^{i,1}, \ldots, x_j^{i,n+1}$.

Each set of points $x_j^i$ is an infinite set in $S$. 
Rainbow Cells to Brower.

Rainbow cell, in $S_j$ with vertices $x_j^{i,1}, \ldots, x_j^{i,n+1}$.

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→ convergent subsequence
Rainbow Cells to Brower.

Rainbow cell, in $J_j$ with vertices $x_j^1, \ldots, x_j^{n+1}$.

Each set of points $x_i^j$ is an infinite set in $S$.

$\rightarrow$ convergent subsequence $\rightarrow$ has limit point.

But $f(x_j^i) < x_j^i$ for all $j$ and $\lim_{j \to \infty} x_j^i = x^*$.

Thus, $(f(x^*))^i \leq x^*_i$ by continuity.

Contradiction.
Rainbow Cells to Brower.

Rainbow cell, in $S_j$ with vertices $x_j^{i,1}, \ldots, x_j^{i,n+1}$.

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$f(x)$ has no fixed point $\implies f(x)_i \geq x_i$ for some $i$. ($\sum_i x_i = 1$).
Rainbow cell, in \( S \) with vertices \( x^j, 1, \ldots, x^j, n+1 \).

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Computing Nash Equilibrium.

PPAD - “Polynomial Parity Argument on Directed Graphs.”
Computing Nash Equilibrium.

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“Graph with an unbalanced node (indegree \(\neq\) outdegree) must have another.”
Computing Nash Equilibrium.

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Exponentially large graph with vertex set $\{0, 1\}^n$. 
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Exponentially large graph with vertex set \( \{0, 1\}^n \).

Circuit given name of graph finds previous, \( P(v) \), and next, \( N(v) \).
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**END OF THE LINE.** Given circuits \( P \) and \( N \) as above, if \( O^n \) is unbalanced node in the graph, find another unbalanced node.
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PPAD is search problems poly-time reducible to END OF LINE.
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**END OF THE LINE.** Given circuits $P$ and $N$ as above, if $O^n$ is an unbalanced node in the graph, find another unbalanced node.

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NASH $\rightarrow$ BROUWER $\rightarrow$ SPERNER $\rightarrow$ END OF LINE $\in$ PPAD.
Other classes.

PPA: “If an undirected graph has a node of odd degree, it must have another.”
Other classes.

PPA: “If an undirected graph has a node of odd degree, it must have another.

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All exist: not \( NP!! \)
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END OF LINE → Piecewise Linear Brouwer
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Reduction:

END OF LINE \( \rightarrow \) Piecewise Linear Brouwer \( \rightarrow \) 3D–Sperner\( \rightarrow \)
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Reduction:
END OF LINE $\rightarrow$ Piecewise Linear Brouwer $\rightarrow$ 3D–Sperner $\rightarrow$ Nash.
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Uh oh. Nash is PPAD-complete.
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Who invented? PapaD and PPAD.
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Who invented? PapaD and PPAD. Perfect together!