Strategic Games.

N players.

Each player has strategy set. $\{S_1, \ldots, S_N\}$.

Vector valued payoff function: $u(s_1, \ldots, s_n)$ (e.g., $\in \Re^N$).

Example:

2 players

Player 1: { **D**efect, **C**ooperate }.
Player 2: { **D**efect, **C**ooperate }.

Payoff:

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(1,1)

Famous because?

	C	**D**
C	(3,3)	(0,5)
D	(5,0)	(.1.1)

What is the best thing for the players to do?

Both cooperate. Payoff $(3,3)$.

If player 1 wants to do better, what does she do?

Defects! Payoff $(5,0)$

What does player 2 do now?

Defects! Payoff $(.1,.1)$.

Stable now!

Nash Equilibrium:
neither player has incentive to change strategy.

Proving Nash.

n players.

Player i has strategy set $\{1, \ldots, m_i\}$.

Payoff function for player i: $u_i(s_1, \ldots, s_n)$ (e.g., $\in \Re^n$).

Mixed strategy for player i: x_i is vector over strategies.

Nash Equilibrium: $x = (x_1, \ldots, x_N)$ where

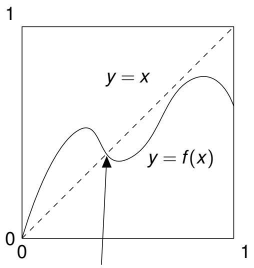
$\forall i \forall x'_i, u_i(x_{-i}; x'_i) \leq u_i(x)$.

What is x? A vector of vectors: vector i is length m_i.
What is $x_{-i}; z$? x with x_i replaced by z.
What does say? No new strategy for player i that is better!

Theorem: There is a Nash Equilibrium.

Brouwer Fixed Point Theorem.

Theorem: Every continuous from from a closed compact convex (c.c.c.) set to itself has a fixed point.



Fixed point!

What is the closed convex set here?

The unit square? Or the unit interval?

Brouwer implies Nash.

The set of mixed strategies x is closed convex set.

That is, $x = (x_1, \ldots, x_n)$ where $|x_i|_1 = 1$.

$\alpha x' + (1-\alpha)x''$ is a mixed strategy.

Define $\phi(x_1, \ldots, x_n) = (z_1, \ldots, z_n)$

where $z_i = \arg\max_{z'_i} \left[u_i(x_{-i;z'_i}) - \|z_i - x_i\|_2^2 \right]$.

Unique minimum as quadratic.
z_i is continuous in x.
Mixed strategy utilities is polynomial of entries of x
with coefficients being payoffs in game matrix.

$\phi(\cdot)$ is continuous on the closed convex set.

Brouwer: Has a fixed point: $\phi(\hat{z}) = \hat{z}$.

Fixed Point is Nash.

$\phi(x_1, \ldots, x_n) = (z_1, \ldots, z_n)$ where
$z_i = \arg\max_{z'_i} \left[u_i(x_{-i;z'_i}) + \|z_i - x_i\|_2^2 \right]$.

Fixed point: $\phi(\hat{z}) = \hat{z}$

If $\hat{z}$ not Nash, there is i, y_i where

$u_i(\hat{z}_{-i}; y_i) > u_i(\hat{(z)}) + \delta$.

Consider $\hat{y}_i = \hat{z}_i + \alpha(y_i - z_i)$.
$u_i(\hat{z_{-i}}; \hat{y}_i) + \|\hat{z}_i - y_i\|^2$?
$u_i(\hat{z}) + \alpha(u_i(\hat{z}) + \delta - u_i(\hat{z})) - \alpha^2 \|\hat{z}_i - y_i\|^2$
$= u_i(\hat{z}) + \alpha\delta - \alpha^2 \|y_i - \hat{z}_i\|^2 > u_i(\hat{z})$.

The last inequality true when $\alpha < \frac{\delta}{\|y_i - z_i\|^2}$.

Thus, $\hat{z}$ not a fixed point!

Thus, fixed point is Nash. □

Sperner's Lemma

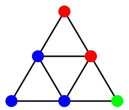
For any $n+1$-dimensional simplex which is subdivided into smaller simplices.

All vertices are colored $\{1,\ldots,n+1\}$.

The coloring is proper if the extremal vertices are differently colored.

Each face only contains the colors of the incident corners.

Lemma: There exist a simplex that has all the colors.



Oops.
Where is multicolored?
Where is multicolored? And now?

By induction!

Proof of Sperner's.

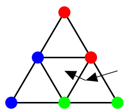
One dimension: Subdivision of $[0,1]$.

Endpoints colored differently.
Odd number of multicolored edges.

Two dimensions.
Consider $(1,2)$ edges.
Separates two regions.
Dual edge connects regions with 1 on right.
Exterior region has excess out-degree:
one more $(1,2)$ than $(2,1)$.
There exist a region with excess in-degree.

$(1,2,1)$ triangle has in-degree=out-degree.
$(2,1,2)$ triangle has in-degree=out-degree.

Must be $(1,2,3)$ triangle.
Must be odd number!



$n+1$-dimensional Sperner.

R: counts "rainbow" cells; has all $n+1$ colors.

Q: counts "almost rainbow" cells; has $\{1,\ldots,n\}$.
Note: exactly one color in $\{1,\ldots,n\}$ used twice.

Rainbow face: $n-1$-dimensional, vertices colored with $\{1,\ldots,n\}$.
X: number of boundary rainbow faces.
Y: number of internal rainbow faces.

Number of Face-Rainbow Cell Adjacencies: $R+2Q = X+2Y$

Rainbow faces on one face of big simplex.
Induction $\implies$ Odd number of rainbow faces.
$\rightarrow X$ is odd $\rightarrow X+2Y$ is odd $R+2Q$ is odd.
R is odd. □

Sperner to Brouwer

Consider simplex:S.
Closed compact sets can be mapped to this.
Let $f(x): S \rightarrow S$.

Infinite sequence of subdivisions: $\mathscr{S}_1, \mathscr{S}_2, \ldots$

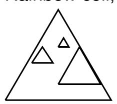
$\mathscr{S}_j$ is subdivision of $\mathscr{S}_{j-1}$. Size of cell $\rightarrow 0$ as $j \rightarrow \infty$.

A coloring of $\mathscr{S}_j$. Recall $\sum_i x_i = 1$ in simplex.
Big simplex vertices $e_j = (0,0,\ldots,1,\ldots,0)$ get j.

For a vertex at x.
Assign smallest i with $f(x)_i < x_i$.
Exists? Yes. $\sum_i f(x)_i = \sum_i x_i$.

Valid? Simplex face is at $x_j = 0$ for opposite j.
Thus $f(x)_j$ cannot be smaller and is not colored j.

Rainbow cell, in $\mathscr{S}_j$ with vertices $x^{j,1},\ldots,x^{j,n+1}$.



Rainbow Cells to Brower.

Rainbow cell, in $\mathscr{S}_j$ with vertices $x^{j,1},\ldots,x_j^{j,n+1}$.

Each set of points x_i^j is an infinite set in S.

$\rightarrow$ convergent subsequence $\rightarrow$ has limit point.
$\rightarrow$ All have same limit point as they get closer together.
x^* is limit point.

$f(x)$ has no fixed point $\implies$ $f(x)_i \geq x_i$ for some i. $(\sum_i x_i = 1)$.

But $f(x^{j,i})_i < x_i^{j,i}$ for all j and
$\lim_{j\rightarrow\infty} x^{j,i} = x^*$.

Thus, $(f(x^*))_i \leq x_i^*$ by continuity. Contradiction. □

Computing Nash Equilibrium.

PPAD - "Polynomial Parity Argument on Directed Graphs."

"Graph with an unbalanced node (indegree $\neq$ outdegree) must have another."

Exponentially large graph with vertex set $\{0,1\}^n$.

Circuit given name of graph finds previous, $P(v)$, and next, $N(v)$.

Sperner: local information gives neighbor.

END OF THE LINE. Given circuits P and N as above, if O^n is unbalanced node in the graph, find another unbalanced node.

PPAD is search problems poly-time reducibile to END OF LINE.

NASH $\rightarrow$ BROUWER $\rightarrow$ SPERNER $\rightarrow$ END OF LINE $\in$ PPAD.

Other classes.

PPA: "If an undirected graph has a node of odd degree, it must have another.

PLS: "Every directed acyclic graph must have a sink."

PPP: "If a function maps n elements to $n-1$ elements, it must have a collision."

All exist: not *NP*!!! Answer is yes. How to find quickly?

Reduction:
END OF LINE $\rightarrow$ Piecewise Linear Brouwer $\rightarrow$ $3D-$Sperner$\rightarrow$ Nash.

Uh oh. Nash is PPAD-complete.

Who invented? PapaD and PPAD. Perfect together!