Today.

Continue markov chain mixing analysis.

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Prove “hard side” of Cheeger.

Analyzing random walks on graph.

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  For d-regular graph: eventually uniform.

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Fix-it-up chappie!

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$$d_2(v_t, \pi) = |A^t e_1 - \pi|^2 \leq \left(\frac{(1+\lambda_2)}{2}\right)^{2t} \leq (1 - \frac{1}{2p(n)})^{2t} \leq \frac{1}{\text{poly}(N)}$$

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$$d_2(v_t, \pi) = |A^t e_1 - \pi|^2 \leq \left(\frac{(1+\lambda_2)}{2}\right)^{2t} \leq (1 - \frac{1}{2p(n)})^{2t} \leq \frac{1}{\text{poly}(N)}$$

Rapidly mixing with big ($\geq \frac{1}{p(n)}$) spectral gap.

Rapid mixing, volume, and surface area..

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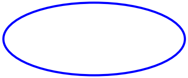

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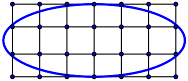

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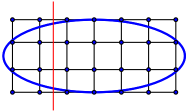

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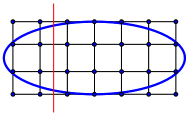

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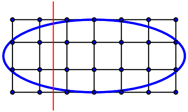

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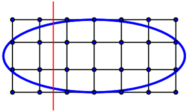

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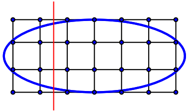

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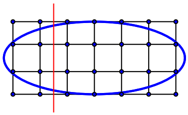

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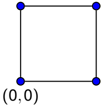

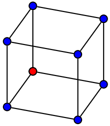

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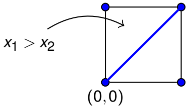

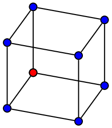

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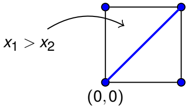

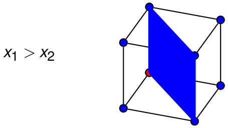

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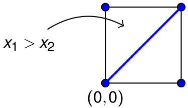

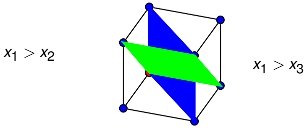

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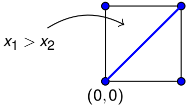

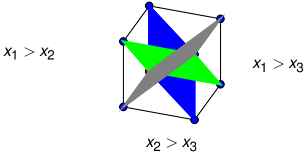

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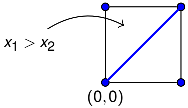

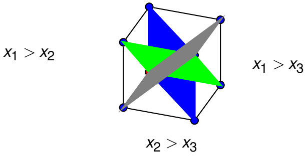

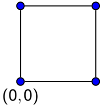
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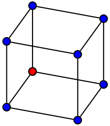

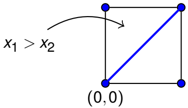
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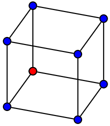

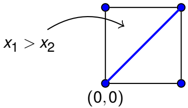
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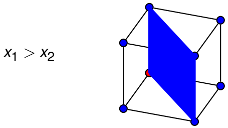
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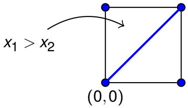
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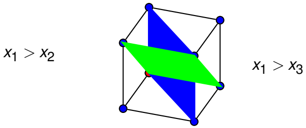
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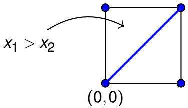

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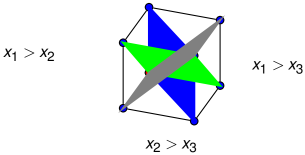

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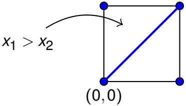
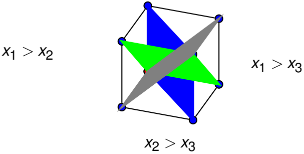
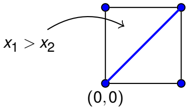
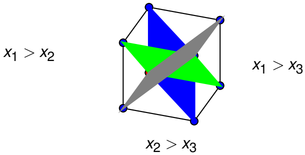




Each order takes $\frac{1}{n!}$ volume.





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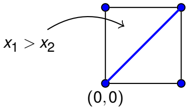

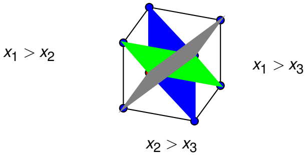

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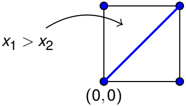

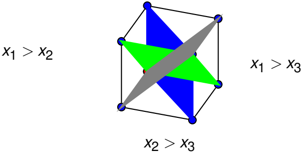

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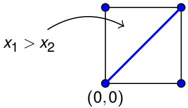

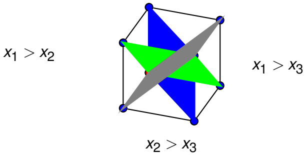

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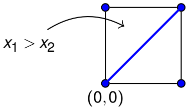

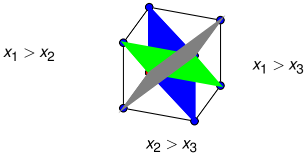

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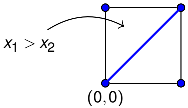

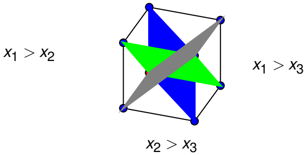

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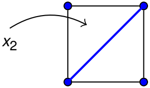

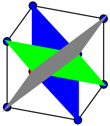

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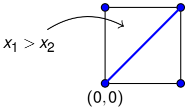

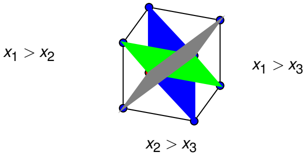

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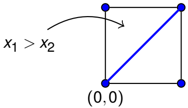

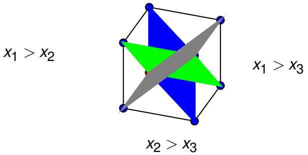

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Do the polynomial dance!!!

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Partial Order Application.

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We don't know. Try all possible thresholds ($n-1$ possibilities), and hope there is a t leading to a good cut!

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Note: Applying the Main Lemma with the 2^{nd} eigenvector v_2, we have $\mu=1-\lambda_2$, and $h(G)\le h(S)\le\sqrt{2(1-\lambda_2)}$. Done!

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$$\begin{aligned}\mathbb{E}_{S\sim D}[min(|S|,|V-S|)] &= \mathbb{E}_{S\sim D}[\sum_i T_i] \\ &= \sum_i \mathbb{E}_{S\sim D}[T_i] \\ &= \sum_i x_i^2\end{aligned}$$

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A common upper bound: $\mathbb{E}[T_{i,j}] = Pr[T_{i,j}=1] \leq |x_i - x_j|(|x_i| + |x_j|)$

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \leq \sqrt{2\mu}$

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Let $T_{i,j} =$ indicator for i, j is cut by $(S, V - S)$

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$$\begin{aligned}
\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V-S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\
&\leq \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \\
&= \frac{1}{2} a \cdot b \\
&\leq \frac{1}{2}\|a\|\|b\|
\end{aligned}$$

Recall $\mu = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}, a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}|x_i| + |x_j|$

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$$
\begin{aligned}
\|a\|^2 &= \sum_{i,j} M_{ij}(x_i - x_j)^2 = \frac{\mu}{n}\sum_{i,j}(x_i - x_j)^2 \\
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\|b\|^2 &= \sum_{i,j} M_{ij}(|x_i| + |x_j|)^2 \\
&\le \sum_{i,j} M_{ij}(2x_i^2 + 2x_j^2) \\
&= 4\sum_i x_i^2
\end{aligned}$$

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Thus $\exists S_i$ such that $h(S_i) \leq \sqrt{2\mu}$, which gives $h(G) \leq \sqrt{2(1-\lambda)}$ $\square$

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Connect to bounding mixing time on Markov Chain.