Today.

Continue markov chain mixing analysis.
Prove “hard side” of Cheeger.

Rapid mixing, volume, and surface area.

Recall volume of convex body.
Grid graph on grid points inside convex body.
Recall Cheeger: $\frac{T}{d} \leq h(G) \leq \sqrt{d}$.
Lower bound expansion $\rightarrow$ lower bounds on spectral gap $\mu$  
Upper bound mixing time: $h(G) = \frac{\text{Surface Area}}{\text{volume}}$  
Isoperimetric inequality: $\text{Vol}_{n-1}(S, \mathcal{B}) = \min \{ \text{Vol}(S), \text{Vol}(\mathcal{B}) \}$  
Edges = surface area, Assume Diam$(P) \leq p(n)$  
$\rightarrow h(G) \geq 1/p(n)$  
$\mu \geq 1/2p(n)^2$  
$\rightarrow O(p(n)^2 \log N)$ convergence for Markov chain on BIG GRAPH.  
$\rightarrow$ Rapidly mixing chain:

Khachiyan’s algorithm for counting partial orders.

Given partial order on $x_1, \ldots, x_n$, 
Sample from uniform distribution over total orders.
Start at an ordering.
Swap random pair and go if consistent with partial order.
Rapidly mixing chain?
Map into $d$-dimensional unit cube.
$x_i < x_j$ corresponds to halfspace (one side of hyperplane) of cube.
“Dimension $i$ = dimension $j$”  
total order is intersection of $n$ halfspaces.
each of volume $\frac{1}{2^n}$  
since each total order is disjoint and together cover cube.

Analyzing random walks on graph.

Start at vertex, go to random neighbor.
For $d$-regular graph: eventually uniform.
if not bipartite. Odd/even step!
How to analyse?
Random Walk Matrix: $M$.
$M$: normalized adjacency matrix.
Symmetric, $\sum_i M[i,j] = 1$.
$M[i,j]$ = probability of going to $j$ from $i$.
Probability distribution at time $t$: $\pi_t$.  
$\pi_t = \frac{1}{N} M^t \pi_0$.  
Each node is average over neighbors.

Fix-it-up chappie!

“Lazy” random walk: With probability 1/2 stay at current vertex.
Evolution Matrix: $\frac{1}{2}M$
Eigenvalues: $\frac{1}{2} \pm \lambda$  
$\frac{1}{2} (I + M) \pi = \frac{1}{2} (\pi + \lambda \pi)$  
$\mu \rightarrow \frac{1}{2}$

Rapid mixing, volume, and surface area.  
Recall volume of convex body. 
Grid graph on grid points inside convex body. 
Side question: Why the same size? Assumed regular graph.

Spectral gap: $1/\mu$  
Uniform distribution: $\pi = [\frac{1}{n}, \ldots, \frac{1}{n}]$.
Distance to uniform: $d_t(\pi, \pi) = \sum |\pi_i - \pi|$.
"Rapidly mixing": $d_t(\pi, \pi) \leq \varepsilon \text{ in } \text{poly}(\log N, \log 1/\varepsilon)$ time.
When is chain rapidly mixing?
Another measure: $d_t(\pi, \pi) = \sum |\pi_i - \pi_j|^2$  
Even more measure: $d_t(\pi, \pi) = |\lambda^{2t} - \pi|_2$  
$d_t(\pi, \pi) = |\lambda^{2t} - \pi|_2$  
$|\lambda^{2t} - \pi|_2 \leq 1 - (1 - 1/p(n)^2)^{2t}$  
$|\lambda^{2t} - \pi|_2 \leq 1 - (1 - 1/p(n)^2)^{loge}$
$d_t(\pi, \pi) = |\lambda^{2t} - \pi|_2$  
Rapidly mixing with big $> \frac{1}{p(n)}$ spectral gap.

Assumed regular graph.
Start at vertex, go to random neighbor.
For $d$-regular graph: eventually uniform.
if ... eigenvalues of value -1: (+1,−1) on two sides.
Side question: Why the same size? Assumed regular graph.

Doh! Why if bipartite?  
Negative eigenvalues of value -1: (+1,−1) on two sides.
Side question: Why the same size? Assumed regular graph.
Summary.

Cheeger Hard Part.

Proof of Main Lemma

WLOG $V = \{1, \ldots, n\}$, $x_1 \leq x_2 \leq \ldots \leq x_n$

Want to show

$\exists i \text{ s.t. } h(S_i) = \frac{1}{d} |E(S, V - S_i)|$ 

$\min(|S_i|, |V - S_i|) \leq \sqrt{2 \mu}$

Probabilistic Argument: Construct a distribution $D$ over $\{S_1, \ldots, S_{n-1}\}$ such that

$\frac{E_S[D(E(S, V - S))]}{E_S[D(\min(|S|, |V - S|))] \leq \sqrt{2\mu}}$

$\rightarrow E_S[D(E(S, V - S)) - \sqrt{2\mu}\min(|S|, |V - S|)] \leq 0$

$\exists S$ 

$\frac{1}{2} E(S, V - S) - \sqrt{2\mu}\min(|S|, |V - S|) \leq 0$

The distribution $D$

WLOG, shift and scale so that $x_1 = 0$, and $x_1^2 + x_2^2 = 1$

Take $f$ from the range $[x_1, x_2]$ with density function $f(t) = 2t$.

Check: $f^2(t)f(t)dt = \int_0^1 2t^2dt + \int_0^1 2t^2dt = x_1^2 + x_2^2 = 1$

$S = \{i : x_i \leq t\}$

Take $D$ as the distribution over $S_1, \ldots, S_{n-1}$ from the above procedure.

Sweep Cut Algorithm

Input: $G = (V, E)$, $x \in \mathbb{R}^V, x \perp 1$

Sort the vertices in non-decreasing order in terms of their values in $x$

WLOG $V = \{1, \ldots, n\}, x_1 \leq x_2 \leq \ldots \leq x_n$

Let $S_i = \{i, \ldots, i\}, i = 1, \ldots, n - 1$

Return $S = \arg\min_i h(S_i)$

Main Lemma: $G = (V, E)$, $d$-regular

$x \in \mathbb{R}^V, x \perp 1, \mu = \frac{\lambda_2}{1 + \lambda_2}$

If $S$ is the output of the sweep cut algorithm, then $h(S) \leq \sqrt{2\mu}$

Note: Applying the Main Lemma with the 2nd eigenvector $v_2$, we have $\mu = 1 - \lambda_2$, and $h(G) \leq h(S) \leq \sqrt{2(1 - \lambda_2)}$.

Done!
Cauchy-Schwarz Inequality
\[ |a \cdot b| \leq \|a\| \|b\|, \text{ as } a \cdot b = \|a\| \|b\| \cos(a, b) \]
Applying with \(a, b \in \mathbb{R}^n\) with \(a_{ij} = \sqrt{M_{ij}} |x_i - x_j|, b_{ij} = \sqrt{M_{ij}} |x_i| + |x_j|\)
\[ \|a\|^2 = \sum_{i,j} M_{ij} (x_i^2) \quad \|b\|^2 = \sum_{i,j} M_{ij} (|x_i| + |x_j|)^2 \leq \sum_{i,j} M_{ij} (2(x_i^2 + x_j^2)) = 4 \sum_i x_i^2 \]

Recall \(\mu = \sum_{i,j} M_{ij} (x_i - x_j)^2 = \frac{1}{n} \sum_i (x_i - \bar{x})^2\)
\[ \|d\|^2 = \sum_{i,j} M_{ij} (x_i - x_j)^2 = \frac{\mu}{n} \sum_i (x_i - \bar{x})^2 \]
\[ = \frac{\mu}{n} \sum_i (x_i^2 + x_j^2) - 2 \sum_i x_i^2 \]
\[ \leq \frac{\mu}{n} \sum_i (x_i^2 + x_j^2) = 2 \mu \sum_{i,j} \frac{1}{2} \]
\[ \|b\|^2 = \sum_{i,j} M_{ij} (|x_i| + |x_j|)^2 \leq \sum_{i,j} M_{ij} (2x_i^2 + 2x_j^2) = 4 \sum_i x_i^2 \]

Summary
Second largest eigenvalue of matrix: \(\lambda_2\).
Bound mixing time.
Connected to "sparse" cuts.
Cheeger: \(\frac{1}{2} \leq h(G) \leq \sqrt{2\mu}\).
Left hand tight: Hypercube.
Right hand tight: Cycle.
Left side proof: produce good Rayleigh quotient vector from sparse cut.
Right hand proof: produce sparse cut from good Rayleigh quotient.
Connect to bounding mixing time on Markov Chain.