Today and for a bit..

Eigenvalues of graphs.

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Through Cuts.

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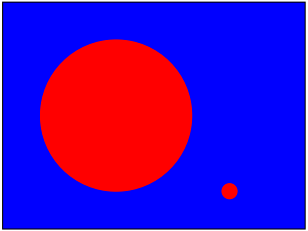
Cheeger's isoperimetric inequality.

Example Problem: clustering.

- Points: documents, dna, preferences.
- Graphs: applications to VLSI, parallel processing, image segmentation.

Image example.

Image Segmentation



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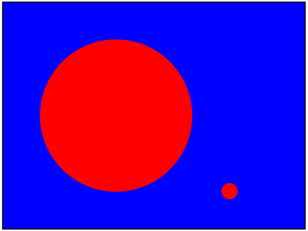

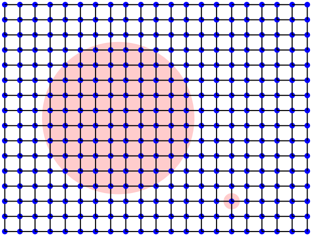

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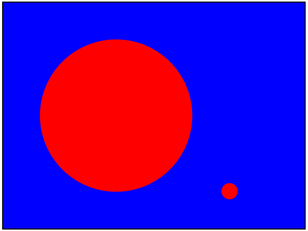
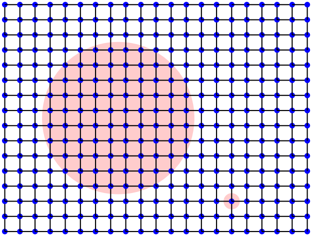


Which region?



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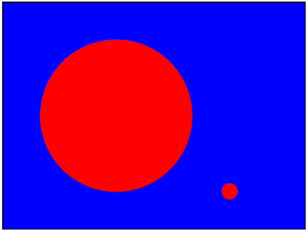

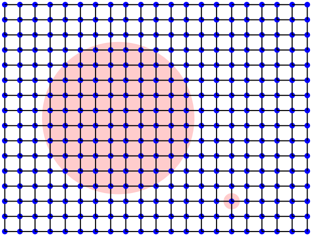

Which region? Normalized Cut: Find S, which minimizes

$$\frac{w(S, \overline{S})}{w(S) \times w(\overline{S})}.$$

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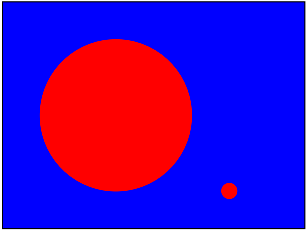

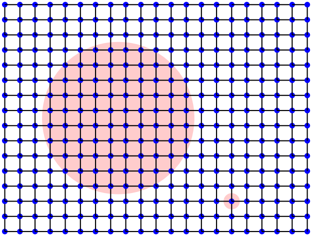

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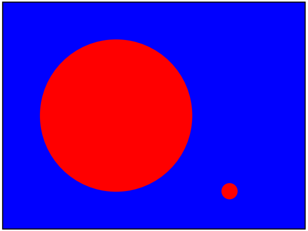

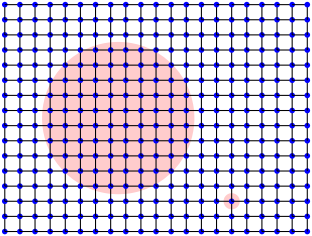

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Either is generally useful!

Edge Expansion/Conductance.

Graph $G = (V, E)$,

Edge Expansion/Conductance.

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Assume regular graph of degree d.

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$h(S) = \frac{|E(S, V-S)|}{d \min |S|, |V-S|}$, $h(G) = \min_S h(S)$

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$\phi(S) = \frac{n|E(S, V-S)|}{d|S||V-S|}$, $\phi(G) = \min_S \phi(S)$

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Note $n \geq \max(|S|, |V| - |S|) \geq n/2$

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Spectra of the graph.

$M = A/d$ adjacency matrix, A

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Distinct eigenvalues

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Distinct eigenvalues $\rightarrow$ orthonormal basis.

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In basis: matrix is diagonal..

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$$M = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Action of M.

v - assigns weights to vertices.

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Eigenvector with highest value? $v = \mathbf{1}$. $\lambda_1 = 1$.

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Proof: Second Eigenvector: $v \perp \mathbf{1}$.

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Connected $\rightarrow$ path from x valued node to lower value.

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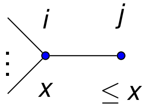

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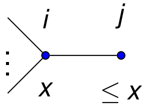

$(Mv)_i \le \frac{1}{d}(x + x \cdots + v_j)$

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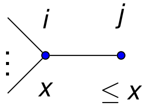

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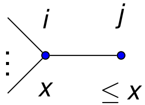

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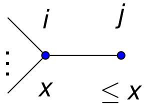

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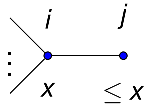

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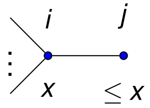

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Therefore $\lambda_2 < 1$. □

Claim: Connected if $\lambda_2 < 1$.
Proof: Assign $+1$ to vertices in one component, $-\delta$ to rest.

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Mv replaces v_i with $\frac{1}{d}\sum_{e=(i,j)} v_j$.

Eigenvector with highest value? $v = \mathbf{1}$. $\lambda_1 = 1$.

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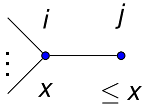

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$x_i = (Mx_i)$

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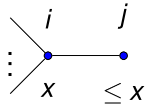

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$x_i = (Mx_i) \implies$ eigenvector with $\lambda = 1$.

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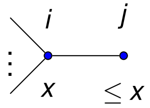

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Choose δ to make $\sum_i x_i = 0$,

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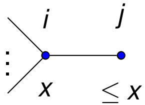
$\rightarrow \exists\, e = (i,j)$, $v_i = x$, $x_j < x$.



$(Mv)_i \leq \frac{1}{d}(x + x \cdots + v_j) < x$.

Therefore $\lambda_2 < 1$. $\square$

Claim: Connected if $\lambda_2 < 1$.
Proof: Assign $+1$ to vertices in one component, $-\delta$ to rest.
$x_i = (Mx_i) \implies$ eigenvector with $\lambda = 1$.
Choose δ to make $\sum_i x_i = 0$, i.e., $x \perp \mathbf{1}$.

Action of M.

v - assigns weights to vertices.

Mv replaces v_i with $\frac{1}{d}\sum_{e=(i,j)} v_j$.

Eigenvector with highest value? $v = \mathbf{1}$. $\lambda_1 = 1$.

$\rightarrow v_i = (M\mathbf{1})_i = \frac{1}{d}\sum_{e\in(i,j)} 1 = 1$.

Claim: For a connected graph $\lambda_2 < 1$.

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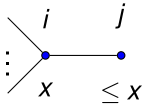

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Rayleigh Quotient

$$\lambda_1 = \max_x \frac{x^T M x}{x^T x}$$

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In basis, M is diagonal.

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Find balanced cut from vector that acheives Rayleigh quotient?

Cheeger's inequality.

Rayleigh quotient.

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$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

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Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

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$\frac{\mu}{2}$

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$\frac{\mu}{2} = \frac{1-\lambda_2}{2}$

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Rayleigh quotient.

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$h(G)$ large

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Hmmm..

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$h(G)$ large $\rightarrow$ well connected

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$h(G)$ small $\rightarrow$ $\lambda_1 - \lambda_2$ small.

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Small cut $\rightarrow$ small eigenvalue gap.

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$\frac{\mu}{2} \leq h(G)$

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Cut S, $|S| \leq |V|/2$.

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Total side endpoints: equal to
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$\frac{v^T M v}{v^T v} = 1 - \frac{|E(S,\overline{S})||V|}{|S||V-S|} \geq 1 - \frac{2|E(S,\overline{S})|}{|S|}$

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Hypercube

$V = \{0, 1\}^d$

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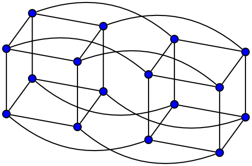

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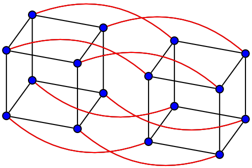

Good cuts? "Coordinate cut": d of them.

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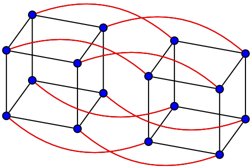

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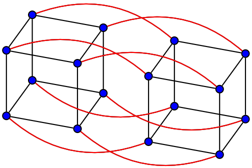

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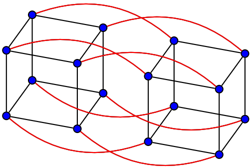

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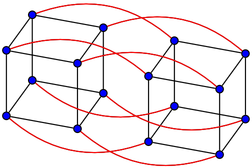

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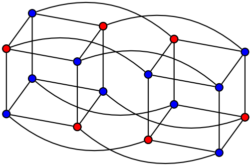

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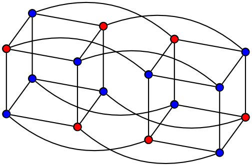

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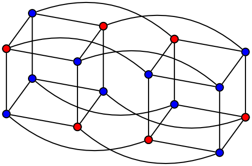

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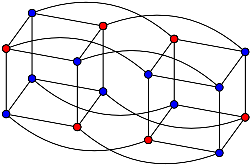

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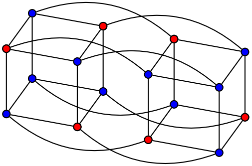

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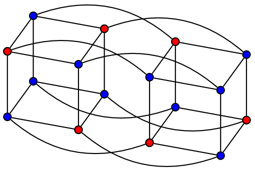

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Worse by a factor of $\sqrt{d}$

Eigenvalues of hypercube.

Anyone see any symmetry?

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Coordinate cuts. +1 on one side, -1 on other.

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Next eigenvectors?

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Eigenvalue: $1-4/d$.

Eigenvalues of hypercube.

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Eigenvalues: $1 - 2k/d$. $\binom{d}{k}$ eigenvectors.

Back to Cheeger.

Coordinate Cuts:

Back to Cheeger.

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Eigenvalue $1 - 2/d$.

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For hypercube: $h(G) = \frac{1}{d}$

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Left hand side is tight.

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Note: hamming weight vector also in first eigenspace.

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Lose "names" in hypercube, find coordinate cut?

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Find coordinate cut?

Eigenvector v maps to line.

Back to Cheeger.

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Eigenvector v maps to line.
Cut along line.

Back to Cheeger.

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Eigenvector algorithm gets a linear combination of coordinate cuts.

Back to Cheeger.

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Left hand side is tight.

Note: hamming weight vector also in first eigenspace.

Lose "names" in hypercube, find coordinate cut?

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Eigenvector v maps to line.
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Something like ball cut.

Back to Cheeger.

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Tight example for Other side of Cheeger?

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Tight example for upper bound for Cheeger.

Eigenvalues of cycle?

Eigenvalues: $\cos \frac{2\pi k}{n}$.

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$(Mx)_i$

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$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right)$

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p - probability distribution.

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Probability distrubtion after choose a random neighbor.

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Start at node 0, probability distribution, $[1, 0, 0, \cdots, 0]$.
Takes $\Omega(n^2)$ to get n steps away.

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Recall drunken sailor.