Today and for a bit..

Eigenvalues of graphs.

Through Cuts.

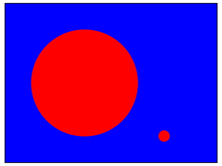
Cheeger's isoperimetric inequality.

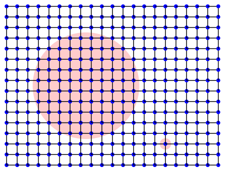
Example Problem: clustering.

- Points: documents, dna, preferences.
- Graphs: applications to VLSI, parallel processing, image segmentation.

Image example.

Image Segmentation





Which region? Normalized Cut: Find S, which minimizes

$$\frac{w(S,\overline{S})}{w(S)\times w(\overline{S})}.$$

Ratio Cut: minimize

$$\frac{w(S,\overline{S})}{w(S)},$$

$w(S)$ no more than half the weight. (Minimize cost per unit weight that is removed.)
Either is generally useful!

Edge Expansion/Conductance.

Graph $G=(V,E)$,

Assume regular graph of degree d.

Edge Expansion.

$h(S) = \frac{|E(S,V-S)|}{d\min|S|,|V-S|}$, $h(G) = \min_S h(S)$

Conductance.

$\phi(S) = \frac{n|E(S,V-S)|}{d|S||V-S|}$, $\phi(G) = \min_S \phi(S)$

Note $n \geq \max(|S|,|V|-|S|) \geq n/2$

$\rightarrow h(G) \leq \phi(G) \leq 2h(G)$

Spectra of the graph.

$M = A/d$ adjacency matrix, A

Eigenvector: a vector v where $Mv = \lambda v$

Real, symmetric.

Claim: Any two eigenvectors with different eigenvalues are orthogonal.

Proof: Eigenvectors: v, v' with eigenvalues λ, λ'.
$v^T M v' = v^T(\lambda' v') = \lambda' v^T v'$
$v^T M v' = \lambda v^T v' = \lambda v^T v$. $\square$
Distinct eigenvalues $\rightarrow$ orthonormal basis.

In basis: matrix is diagonal..

$$M = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Action of M.

v - assigns weights to vertices.

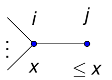
Mv replaces v_i with $\frac{1}{d}\sum_{e=(i,j)} v_j$.

Eigenvector with highest value? $v = \mathbf{1}$. $\lambda_1 = 1$.

$\rightarrow v_i = (M\mathbf{1})_i = \frac{1}{d}\sum_{e\in(i,j)} 1 = 1$.

Claim: For a connected graph $\lambda_2 < 1$.

Proof: Second Eigenvector: $v \perp \mathbf{1}$. Max value x.
Connected $\rightarrow$ path from x valued node to lower value.
$\rightarrow \exists\ e = (i,j)$, $v_i = x$, $x_j < x$.



$(Mv)_i \le \frac{1}{d}(x + x \cdots + v_j) < x$.

Therefore $\lambda_2 < 1$. □

Claim: Connected if $\lambda_2 < 1$.
Proof: Assign $+1$ to vertices in one component, $-\delta$ to rest.
$x_i = (Mx_i) \implies$ eigenvector with $\lambda = 1$.
Choose δ to make $\sum_i x_i = 0$, i.e., $x \perp \mathbf{1}$. □

Rayleigh Quotient

$\lambda_1 = \max_x \frac{x^T M x}{x^T x}$

In basis, M is diagonal.

Represent x in basis, i.e., $x_i = x \cdot v_i$.

$xMx = \sum_i \lambda_i x_i^2 \le \lambda_1 \sum_i x_i^2 \lambda = \lambda x^T x$

Tight when x is first eigenvector. □

Rayleigh quotient.
$\lambda_2 = \max_{x\perp\mathbf{1}} \frac{x^T M x}{x^T x}$.

$x \perp 1 \leftrightarrow \sum_i x_i = 0$.

Example: $-1/+1$ Indicator vector for balanced cut, S is one such vector.

Rayleigh quotient is $\frac{2|E(S,S)|}{2|S|} = h(S)$.

Rayleigh quotient is less than $h(S)$ for any balanced cut S.

Find balanced cut from vector that acheives Rayleigh quotient?

Cheeger's inequality.

Rayleigh quotient.
$\lambda_2 = \max_{x\perp\mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S,|S|\le|V|/2} \frac{|E(S,V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.
$h(G)$ large $\rightarrow$ well connected $\rightarrow \lambda_1 - \lambda_2$ big.
Disconnected $\lambda_2 = \lambda_1$.
$h(G)$ small $\rightarrow \lambda_1 - \lambda_2$ small.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \le h(G)$

Cut S, $|S| \le |V|/2$.
$i \in S: v_i = |V| - |S|$, $i \in \overline{S}: v_i = -|S|$.

$\sum_i v_i = |S|(|V|-|S|) - |S|(|V|-|S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V|-|S|)^2 + |S|^2(|V|-|S|) = |S|(|V|-|S|)(|V|)$.

$v^T M v = \frac{1}{d}\sum_{e=(i,j)} x_i x_j$.

Total side endpoints: equal to
$v^T v - |E(S,S)||S|^2 - |E(S,S)|(|V-S|)^2$

Diff. side endpoints: $-|S|(|V|-|S|)$ each or $-2|E(S,S)||S|(|V|-|S|)$

$v^T M v = v^T v - (2|E(S,S)||V|^2)$

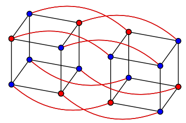
$\frac{v^T M v}{v^T v} = 1 - \frac{|E(S,\overline{S})||V|}{|S||V-S|} \ge 1 - \frac{2|E(S,\overline{S})|}{|S|}$

$\lambda_2 \ge 1 - 2h(S) \rightarrow h(G) \ge \frac{1-\lambda_2}{2}$

Hypercube

$V = \{0,1\}^d$ $\quad (x,y) \in E$ when x and y differ in one bit.

$|V| = 2^d$ $|E| = d2^{d-1}$.



Good cuts? "Coordinate cut": d of them.
Edge expansion: $\frac{2^{d-1}}{d2^{d-1}} = \frac{1}{d}$

Ball cut: All nodes within $d/2$ of node, say $00\cdots0$.
Vertex cut size: $\binom{d}{d/2}$ bit strings with $d/2$ 1's.

$\approx \frac{2^d}{\sqrt{d}}$

Vertex expansion: $\approx \frac{1}{\sqrt{d}}$.

Edge expansion: $d/2$ edges to next level. $\approx \frac{1}{2\sqrt{d}}$

Worse by a factor of $\sqrt{d}$

Eigenvalues of hypercube.

Anyone see any symmetry?

Coordinate cuts. +1 on one side, -1 on other.

$(Mv)_i = (1 - 2/d)v_i$.

Eigenvalue $1 - 2/d$. d Eigenvectors. Why orthogonal?

Next eigenvectors?

Delete edges in two dimensions.

Four subcubes: bipartite. Color ± 1

Eigenvalue: $1 - 4/d$. $\binom{d}{2}$ eigenvectors.

Eigenvalues: $1 - 2k/d$. $\binom{d}{k}$ eigenvectors.

Back to Cheeger.

Coordinate Cuts:
Eigenvalue $1-2/d$. d Eigenvectors.

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

For hypercube: $h(G) = \frac{1}{d}$ $\lambda_1 - \lambda_2 = 2/d$.
Left hand side is tight.

Note: hamming weight vector also in first eigenspace.

Lose "names" in hypercube, find coordinate cut?

Find coordinate cut?

Eigenvector v maps to line.
Cut along line.

Eigenvector algorithm gets a linear combination of coordinate cuts.

Something like ball cut.

Find coordinate cut?

Cycle

Tight example for Other side of Cheeger?

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Cycle on n nodes.

Will show other side of Cheeger is tight.

Edge expansion:Cut in half.
$|S| = n/2$, $|E(S,\overline{S})| = 2$
$\rightarrow h(G) = \frac{2}{n}$.

Show eigenvalue gap $\mu \le \frac{1}{n^2}$.

Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^T Mx}{x^T x}$ close to 1.

Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^T Mx}{x^T x}$ close to 1.

$$x_i = \begin{cases} i - n/4 & \text{if } i \le n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$$

Hit with M.

$$(Mx)_i = \begin{cases} -n/4 + 1/2 & \text{if } i = 1, n \\ n/4 - 1 & \text{if } i = n/2 \\ x_i & \text{otherwise} \end{cases}$$

$\rightarrow x^T Mx = x^T x(1 - O(\frac{1}{n^2}))$ $\quad \rightarrow \lambda_2 \ge 1 - O(\frac{1}{n^2})$

$\mu = \lambda_1 - \lambda_2 = O(\frac{1}{n^2})$

$h(G) = \frac{2}{n} = \Theta(\sqrt{\mu})$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Tight example for upper bound for Cheeger.

Eigenvalues of cycle?

Eigenvalues: $\cos\frac{2\pi k}{n}$.

$x_i = \cos\frac{2\pi ki}{n}$

$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right) = 2\cos\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi ki}{n}\right)$

Eigenvalue: $\cos\frac{2\pi k}{n}$.

Eigenvalues:
vibration modes of system.
Fourier basis.

Random Walk.

p - probability distribution.

Probability distrubtion after choose a random neighbor.
Mp.

Converge to uniform distribution.

Power method: $M^t x$ goes to highest eigenvector.

$M^t x = a_1 \lambda_1^t v_1 + a_2 \lambda_2 v_2 + \cdots$

$\lambda_1 - \lambda_2$ - rate of convergence.

$\Omega(n^2)$ steps to get close to uniform.

Start at node 0, probability distribution, $[1, 0, 0, \cdots, 0]$.
Takes $\Omega(n^2)$ to get n steps away.

Recall drunken sailor.