Today and for a bit..

Eigenvalues of graphs.
Through Cuts.
Cheeger's isoperimetric inequality.

Example Problem: clustering.
▶ Points: documents, dna, preferences.
▶ Graphs: applications to VLSI, parallel processing, image segmentation.

Image example.

Image Segmentation

Which region? Normalized Cut: Find $S$, which minimizes

$$\frac{w(S, \overline{S})}{w(S) \times w(\overline{S})}$$

Ratio Cut: minimize

$$\frac{w(S, \overline{S})}{w(S)}, \frac{w(S)}{w(\overline{S})}$$

w(S) no more than half the weight. (Minimize cost per unit weight that is removed.) Either is generally useful!

Edge Expansion/Conductance.

Graph $G = (V, E)$.
Assume regular graph of degree $d$.

Edge Expansion.

$$h(S) = \frac{|E(S, \overline{S})|}{d \times |S| \times |\overline{S}|}, h(G) = \min_S h(S)$$

Conductance.

$$\phi(S) = \frac{n \times |E(S, \overline{S})|}{d \times |S| \times |\overline{S}|}, \phi(G) = \min_S \phi(S)$$

Note $n \geq \max(|S|, |V| - |S|) \geq n/2$

$$\rightarrow h(G) \leq \phi(G) \leq 2h(G)$$

Either is generally useful!

Spectra of the graph.

$$M = \frac{A}{d}$$ adjacency matrix, $A$

Eigenvector: a vector $v$ where $Mv = \lambda v$
Real, symmetric.

Claim: Any two eigenvectors with different eigenvalues are orthogonal.

Proof: Eigenvectors: $v, v'$ with eigenvalues $\lambda, \lambda'$.

$$v^T M v' = v^T \frac{A}{d} v' = \lambda v^T v'$$

$$v^T M v = \lambda v^T v' = \lambda v^T v.$$  \[\square\]

Distinct eigenvalues $\rightarrow$ orthonormal basis.
In basis: matrix is diagonal.

$$M = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}$$
Action of $M$.

$v$ - assigns weights to vertices.

$Mv$ replaces $v$ with $\frac{1}{n} \sum_{i \neq j} v_i$.

Eigenvector with highest value? $\lambda = 1$.

$\lambda_1 \leq 1$.

Claim: For a connected graph $\lambda_2 < 1$.

Proof: Second Eigenvector: $v \perp 1$. Max value $x$.

Connected $\rightarrow$ path from $x$ valued node to lower value.

$\Rightarrow \exists \mathbf{v} = (i, j), v_i = x, v_j < x$.

\[
\sum_{i \neq j} v_i \leq \frac{1}{2} (x + x + \ldots + v_j) < x.
\]

Claim: Connected if $\lambda_2 < 1$.

Proof: Assign $\pm 1$ to vertices in one component, $-\delta$ to rest.

Choose $\delta$ to make $\sum x_i = 0$, i.e., $x \perp 1$.

Rayleigh Quotient

In basis, $M$ is diagonal.

$\lambda_2 = \max_{x \perp 1} \frac{x^T Mx}{x^T x}$

$\sum_{i} \lambda_i x_i^2 \leq \lambda_1 \sum_{i} x_i^2 \Rightarrow \lambda x^T x$

Rayleigh quotient.

$\lambda_2 = \max_{x, \mu} \frac{x^T Mx}{x^T x - \mu x^T x}$

Rayleigh quotient.

Eigenvector with highest value?

$\lambda_2 < \lambda_1$.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{1}{2} \leq h(G)$

Cut $S, |S| \leq |V|/2$.

$i \in S \Rightarrow v_i = |V| - |S|, i \in \overline{S}, v_i = -|S|$

$\sum v_i = |S||V| - |S||S| - |S||V||V| - 0$

$\Rightarrow v \perp 1$.

$\lambda_2 \geq |S| - |S|^2 + |S^2||V||S^2 - |S||V^2 - |S||V||V|$

$\lambda_2 \geq 1 - \frac{|E(S, S)| |V|}{|S||V||V|}$

Total side endpoints: equivalent to $v^T v - |E(S, S)||V| - |E(S, S)| |V - S|^2$

Diff. side endpoints: $|S|(|V| - |S|)$ each or $-2|E(S, S)||S||V - S|$.

$\sum v_i = \frac{1}{2} |E(S, S)| |V|$

$\frac{\lambda_2 Mv}{v^T v} = 1 - \frac{|E(S, S)| |V|}{|S||V||V|} \geq 1 - \frac{2|E(S, S)| |V|}{|S||V||V|}$

$\lambda_2 \geq 1 - 2h(S) \Rightarrow h(G) \geq \frac{\lambda_2}{2}$

Worse by a factor of $\sqrt{3}$

Rayleigh Quotient

$\lambda_2 = \max_{x \perp 1} \frac{x^T Mx}{x^T x}$

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$\sum_{i} \lambda_i x_i^2 \leq \lambda_1 \sum_{i} x_i^2 \lambda_1 x_i^2\lambda_1 x_i^2$

Tight when $x$ is first eigenvector.

Rayleigh quotient.

$\lambda_2 = \max_{x, \mu} \frac{x^T Mx}{x^T x - \mu x^T x}$

Rayleigh quotient.

$\lambda_2 < \lambda_1$.

Cheeger’s inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x, \mu} \frac{x^T Mx}{x^T x - \mu x^T x}$

Eigenvector gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{\lambda(S)}{2|E(S, S)||V|}$

$\frac{1}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2^2)} = \sqrt{\frac{2}{3}}$

Hmmm.. Connected $\lambda_2 < \lambda_1$.

$\mu$ large $\rightarrow$ well connected $\rightarrow \lambda_1 - \lambda_2$ big.

Disconnected $\lambda_2 = \lambda_1$.

$h(G)$ small $\rightarrow \lambda_1 - \lambda_2$ small.

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Worse by a factor of $\sqrt{3}$

Hypercube

$V = \{0, 1\}^d$ $(x, y) \in E$ when $x$ and $y$ differ in one bit.

$|V| = 2^d |E| = d2^{d-1}$.

Good cuts? Coordinate cuts: $d$ of them.

Edge expansion: $\frac{d}{2^{d-1}} = \frac{1}{2}$

Ball cut: All nodes within $d/2$ of node, say $00 \cdots 0$.

Vertex cut size: $\binom{d}{2} 2$ bit strings with $d/2$ 1’s.

$= \frac{d}{2^{d-1}}$

Edge expansion: $\frac{d}{2}$ edges to next level. $\approx \frac{1}{2^{d-1}}$

Eigenvector of hypercube.

$\lambda_2 = \max_{x \perp 1} \frac{x^T Mx}{x^T x}$

$\sum_{i} \lambda_i x_i^2 \leq \lambda_1 \sum_{i} x_i^2 \lambda_1 x_i^2\lambda_1 x_i^2$

Tight when $x$ is first eigenvector.

Rayleigh quotient.

$\lambda_2 = \max_{x, \mu} \frac{x^T Mx}{x^T x - \mu x^T x}$

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Disconnected $\lambda_2 = \lambda_1$.

$h(G)$ small $\rightarrow \lambda_1 - \lambda_2$ small.

Hyperslices

$\lambda_2 = \max_{x \perp 1} \frac{x^T Mx}{x^T x}$

In basis, $M$ is diagonal.

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Cheeger’s inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp 1} \frac{x^T Mx}{x^T x}$

Eigenvector gap: $\mu = \lambda_1 - \lambda_2$.

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$h(G)$ small $\rightarrow \lambda_1 - \lambda_2$ small.
Back to Cheeger.

Coordinate Cuts:
Eigenvalue $1 - 2/d$. d Eigenvectors.
$\frac{h}{2} = \frac{1}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)} = \sqrt{2/\mu}$
For hypercube: $h(G) = \frac{1}{2} \lambda_1 - \lambda_2 = 2/d$.
Left hand side is tight.
Note: hamming weight vector also in first eigenspace.
Lose “names” in hypercube, find coordinate cut?
Find coordinate cut?
Eigenvalue $cos \frac{2\pi k}{n}$.
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Eigenvalues:

Cycle

Tight example for Other side of Cheeger?
$\frac{h}{2} = \frac{1}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)} = \sqrt{2/\mu}$
Cycle on $n$ nodes.
Will show other side of Cheeger is tight.
Edge expansion: Cut in half. $|S| = n/2, |E(S,S)| = 2$
$\rightarrow h(G) = \frac{2}{n}$.
Show eigenvalue gap $\mu \leq \frac{1}{\sqrt{n}}$.
Find $x \perp 1$ with Rayleigh quotient $\frac{x^TMx}{x^Tx}$ close to 1.

Eigenvalues of cycle?

Eigenvalues: $cos \frac{2\pi k}{n}$.

$\begin{cases} i - n/4 & \text{if } i \leq n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$
Hit with $M$.

Random Walk.

$\rho$ - probability distribution.
Probability distribution after choose a random neighbor. $M\rho$.
Converge to uniform distribution.
Power method: $M^k x$ goes to highest eigenvector.
$M^k x = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \cdots$
$\lambda_1 - \lambda_2$ - rate of convergence.
$\Omega(\sqrt{n})$ steps to get close to uniform.
Start at node 0, probability distribution, $[1,0,0,\cdots,0]$.
Takes $\Omega(n^2)$ to get $n$ steps away.
Recall drunken sailor.