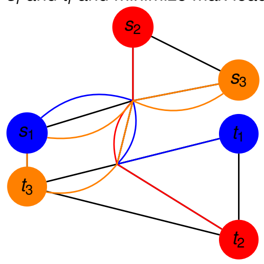
CS270: Lecture 2.

Admin:
Check Piazza.
Today:

- Finish Path Routing.
- ????

Path Routing.

Given $G=(V,E)$, $(s_1,t_1),\ldots,(s_k,t_k)$, find a set of k paths connecting s_i and t_i and minimize max load on any edge.

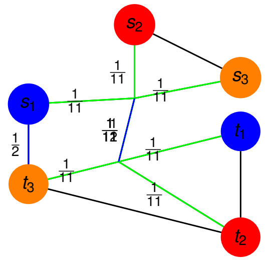


~~Value: 3~~

Value: 2

Another problem.

Given $G=(V,E)$, $(s_1,t_1),\ldots,(s_k,t_k)$, find a set of k paths assign one unit of "toll" to edges to maximize total toll for connecting pairs.



Assign $\frac{1}{11}$ on each of 11 edges.
Toll paid: $\frac{3}{11} + \frac{3}{11} + \frac{3}{11} = \frac{9}{11}$

Can we do better?

Assign $1/2$ on these two edges.
Toll paid: $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$

Terminology

Routing: Paths $p_1,p_2,\ldots,p_k$, p_i connects s_i and t_i.

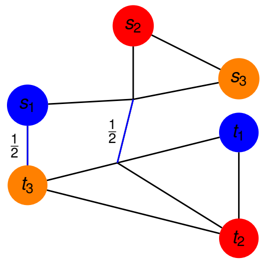
Congestion of edge, e: $c(e)$
  number of paths in routing that contain e.

Congestion of routing: maximum congestion of any edge.

Find routing that minimizes congestion (or maximum congestion.)

Toll problem.

Given $G=(V,E)$, $(s_1,t_1),\ldots,(s_k,t_k)$, find a set of k paths assign one unit of "toll" to edges to maximize total toll for connecting pairs.



Assign $\frac{1}{11}$ on each of 11 edges.
Total toll: $\frac{3}{11}+\frac{3}{11}+\frac{3}{11}=\frac{9}{11}$

Can we do better?

Assign $1/2$ on these two edges.
Total toll: $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$

Toll: Terminology.

$d(e)$ - toll assigned to edge e.
Note: $\sum_e d(e)=1$. $d(p)$ - total toll assigned to path p.
$d(u,v)$ - total assigned to shortest path between u and v.
$d(x)$ - ~~polymorpic~~ polymorphic
  x could be edge, path, or pair.

Toll is lower bound on Path Routing.

From before:
Max bigger than minimum weighted average:
$\max_e c(e) \geq \sum_e c(e)d(e)$
Total length is total congestion:
$\sum_e c(e)d(e) = \sum_i d(p_i)$
Each path, p_i, in routing has length $d(p_i) \geq d(s_i, t_i)$.

$$\max_e c(e) \geq \sum_e c(e)d(e) = \sum_i d(p_i) \geq \sum_i d(s_i, t_i).$$

A toll solution is lower bound on any routing solution.

Any routing solution is an upper bound on a toll solution.

Algorithm.

Assign tolls according to routing.

How to route? Shortest paths!

Assign routing according to tolls.

How to assign tolls? Higher tolls on congested edges.

Toll: $d(e) =\propto 2^{c(e)}$. $\sum_e d(e) = 1$.

Equilibrium:
The shortest path routing has has $d(e) \propto 2^{c(e)}$.

"The routing is stable, the tolls are stable."

Routing: each path p_i in routing is a shortest path w.r.t $d(\cdot)$

Tolls: ...where $d(e)$ is defined w.r.t. to current routing.
Subtlety here due to $\sum_e d(e) = 1$.

How good is equilibrium?

Path is routed along shortest path and $d(e) = \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}}$.

For e with $c(e) \leq c_{max} - 2\log m$; $2^{c(e)} \leq 2^{c_{max} - 2\log m} = \frac{2^{c_{max}}}{m^2}$.

$$\begin{aligned}
c_{opt} &\geq \sum_i d(s_i, t_i) = \sum_e d(e)c(e) \\
&= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e) = \frac{\sum_e 2^{c(e)} c(e)}{\sum_e 2^{c(e)}} \quad \text{Let } c_t = c_{max} - 2\log m. \\
&\geq \frac{\sum_{e:c(e)>c_t} 2^{c(e)} c(e)}{\sum_{e:c(e)>c_t} 2^{c(e)} + \sum_{e:c(e)\leq c_t} 2^{c(e)}} \\
&\geq \frac{(c_t)\sum_{e:c(e)>c_t} 2^{c(e)}}{(1+\frac{1}{m})\sum_{e:c(e)>c_t} 2^{c(e)}} \\
&\geq \frac{(c_t)}{1+\frac{1}{m}} = \frac{c_{max} - 2\log m}{(1+\frac{1}{m})}
\end{aligned}$$

Or $c_{max} \leq (1+\frac{1}{m})c_{opt} + 2\log m$.
(Almost) within additive term of $2\log m$ of optimal!

Getting to equilibrium.

Maybe no equilibrium!

Approximate equilibrium:

Each path is routed along a path with length
within a factor of 3 of the shortest path and $d(e) \propto 2^{c(e)}$.

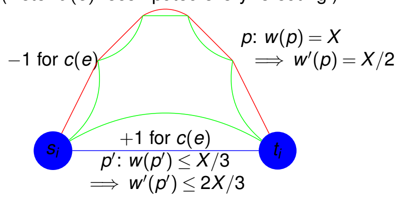
Lose a factor of three at the beginning.
$c_{opt} \geq \sum_i d(s_i, t_i) \geq \frac{1}{3} \sum_e d(p_i) = \frac{1}{3} \sum_e d(e)c(e)$

We obtain $c_{max} = 3(1+\frac{1}{m})c_{opt} + 2\log m$.

This is worse!
What do we gain?

An algorithm!

Repeat: reroute any path that is off by a factor of 3.
(Note: $d(e)$ recomputed every rerouting.)



Potential function: $\sum_e w(e)$, $w(e) = 2^{c(e)}$
Moving path:
Divides $w(e)$ along long path (with $w(p)$ of X) by two.
Multiplies $w(e)$ along shorter ($w(p) \leq X/3$) path by two.

$-\frac{X}{2} + \frac{X}{3} = -\frac{X}{6}$.

Potential function decreases. $\implies$ termination and existence.

Tuning...

Replace $d(e) = (1+\varepsilon)^{c(e)}$.

Replace factor of 3 by $(1+2\varepsilon)$

$c_{max} \leq (1+2\varepsilon)c_{opt} + 2\log m/\varepsilon$.. (Roughly)

Fractional paths?

Revisit Equilibrium.

Solution Pair: $(\{p_i\}, d())$.

Toll Solution Value: $\sum_i d(s_i, t_i)$. Path Routing Value: $\max_e c(e)$.

Toll player assigns toll on only maximally congested edges.
Routing player routes on only cheapest paths.

Routing R uses shortest paths. Summation Switch

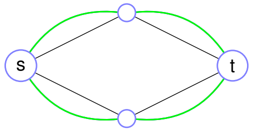
$d(e) \geq 0$. Only Toll on max congestion. $\sum_e d(e) = 1$

$$\begin{aligned}\sum_i d(s_i,t_i) &= \sum_i d(p_i)\\ &= \sum_e c(e)d(e)\\ &= \sum_{e:d(e)>0} c(e)d(e)\\ &= \sum_{e:d(e)>0} d(e)(\max_e c(e))\\ &= \max_e c(e)\end{aligned}$$

Any routing solution value $\geq$ **Any** toll solution value.
Both these solutions are optimal!!!!! Complementary slackness.

Why all the mess before? To get an algorithm!

Algorithm:exact?



Not shortest when tolls on top.
Hmmm...
Uh oh?
Route half a unit on both!

Hey! Fractional!

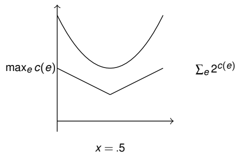
Use previous algorithms but route two paths between each pair.

Half integral!

Optimality: $(3)C_{\max} + 2\log m/2$.

Additive factor shrinking!
The 3 can be made $(1+\varepsilon)$ using different base!

Geometrical view.



Smooth: use $\sum_e 2^{c(e)}$ as a proxy for $\max_e c(e)$.
Minimize new function.
Gradient descent.
Stepsize=1. Back and forth!
Stepsize=.5. Back and forth ...but closer to minimum.

Wrap up.

Dueling players:
Toll player raises tolls on congested edges.
Congestion player avoids tolls.

Converges to near optimal solution!

A lower bound is “necessary” (natural),
and helpful (mysterious?)!

Geometric View: Smooth. Gradient Descent. Stepsize.