Welcome back...

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Metric spaces.

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Which are metric spaces?

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(B) X from R^d and $d(\cdot,\cdot)$ is squared Euclidean distance.

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Approximate metric using a tree.

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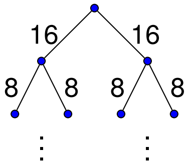

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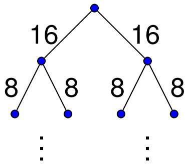

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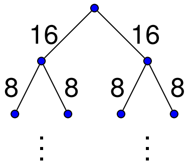

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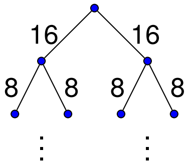

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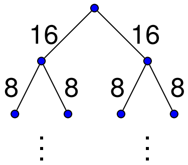

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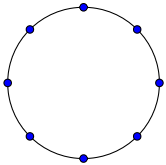

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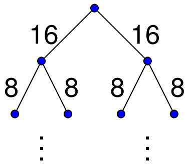

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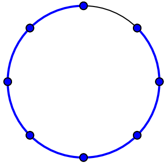

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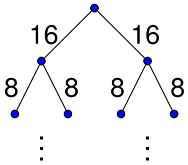

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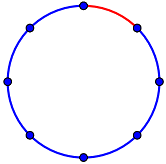

Distance 1 goes to $n-1$!

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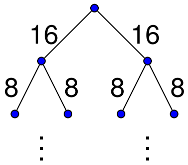

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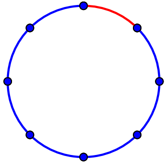

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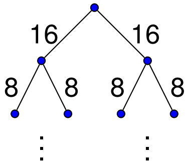

Probabilistic Tree embedding.
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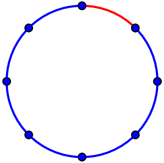

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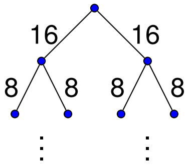

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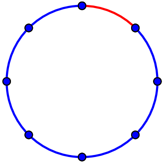

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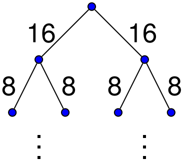

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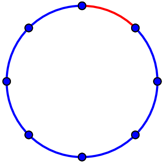

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For cycle, remove a random edge get a tree.

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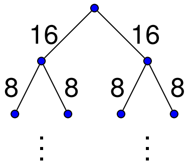

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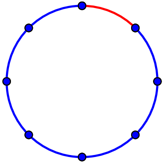

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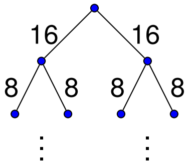

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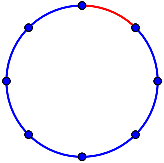

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Stretch of edge: $\frac{n-1}{n} \times 1 + \frac{1}{n} \times (n-1)$

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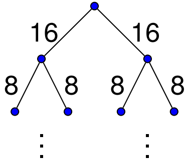

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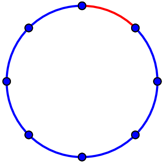

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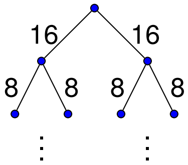

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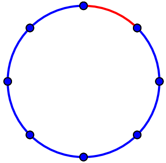

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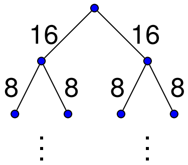

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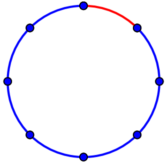

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General metrics?

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Algorithm

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Tree has internal node for each level of call.

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Analysis: idea

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random permutation to deal with this

Analysis: (x, y)

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We gave an algorithm that produces a distribution of trees.

The expected stretch of any pair is $O(\log n)$.

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$\rightarrow O(\log n)$ approximation.

And Now For Something...

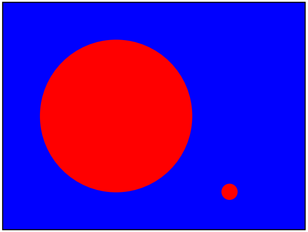
Completely Different.

Example Problem: clustering.

- Points: documents, dna, preferences.
- Graphs: applications to VLSI, parallel processing, image segmentation.

Image example.

Image Segmentation



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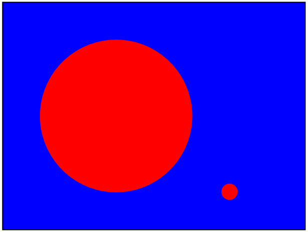

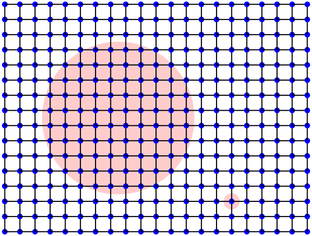

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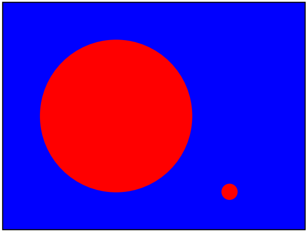
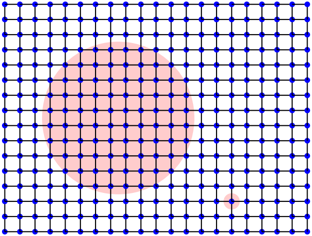


Which region?



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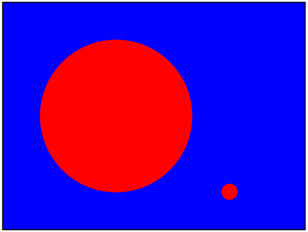

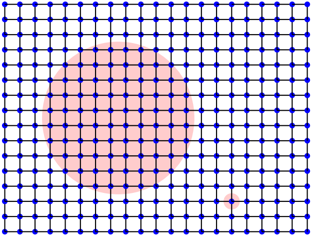

Which region?   Normalized Cut: Find S, which minimizes

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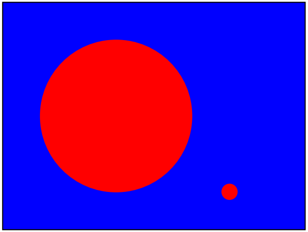

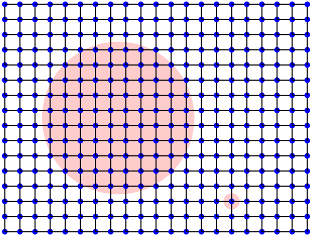

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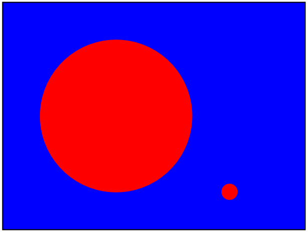 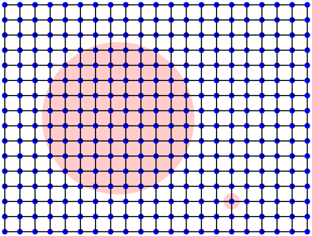

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Either is generally useful!

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Distinct eigenvalues $\rightarrow$ orthonormal basis.

Spectra of the graph.

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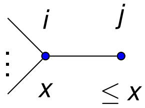

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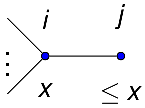

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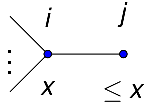

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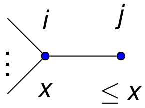

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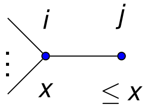

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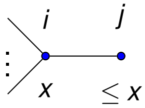

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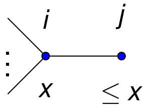

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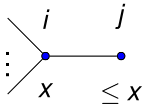

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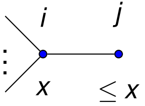

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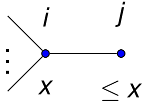

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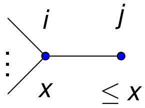

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$$\lambda_1 = \max_x \frac{x^T M x}{x^T x}$$

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Find balanced cut from vector that acheives Rayleigh quotient?

Cheeger's inequality.

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Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2}$

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

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Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G)$

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

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$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)}$

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

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Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

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Hmmm..

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

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$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.

$h(G)$ large

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.

$h(G)$ large $\rightarrow$ well connected

Cheeger's inequality.

Rayleigh quotient.
$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \le |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \le h(G) \le \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.
$h(G)$ large $\rightarrow$ well connected $\rightarrow$ $\lambda_1 - \lambda_2$ big.

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.

$h(G)$ large $\rightarrow$ well connected $\rightarrow$ $\lambda_1 - \lambda_2$ big.

Disconnected

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.

$h(G)$ large $\rightarrow$ well connected $\rightarrow \lambda_1 - \lambda_2$ big.

Disconnected $\lambda_2 = \lambda_1$.

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.

$h(G)$ large $\rightarrow$ well connected $\rightarrow$ $\lambda_1 - \lambda_2$ big.

Disconnected $\lambda_2 = \lambda_1$.

$h(G)$ small

Cheeger's inequality.

Rayleigh quotient.

$\lambda_2 = \max_{x \perp \mathbf{1}} \frac{x^T M x}{x^T x}$.

Eigenvalue gap: $\mu = \lambda_1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Hmmm..

Connected $\lambda_2 < \lambda_1$.
$h(G)$ large $\rightarrow$ well connected $\rightarrow \lambda_1 - \lambda_2$ big.
Disconnected $\lambda_2 = \lambda_1$.
$h(G)$ small $\rightarrow \lambda_1 - \lambda_2$ small.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|)$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Same side endpoints: like $v^T v$.

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S: v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

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Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S: v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Same side endpoints: like $v^T v$.

Different side endpoints: $-|S|(|V| - |S|)$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Same side endpoints: like $v^T v$.

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Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Same side endpoints: like $v^T v$.

Different side endpoints: $-|S|(|V| - |S|)$

$v^T M v = v^T v - (2|E(S,S)||S|(|V| - |S|)$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Same side endpoints: like $v^T v$.

Different side endpoints: $-|S|(|V| - |S|)$

$v^T M v = v^T v - (2|E(S,S)||S|(|V| - |S|)$

$\frac{v^T M v}{v^T v} = 1 - \frac{2|E(S,\overline{S})|}{|S|}$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S : v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Same side endpoints: like $v^T v$.

Different side endpoints: $-|S|(|V| - |S|)$

$v^T M v = v^T v - (2|E(S,S)||S|(|V| - |S|)$

$\frac{v^T M v}{v^T v} = 1 - \frac{2|E(S,\overline{S})|}{|S|}$

$\lambda_2 \geq 1 - 2h(S)$

Easy side of Cheeger.

Small cut $\rightarrow$ small eigenvalue gap.

$\frac{\mu}{2} \leq h(G)$

Cut S. $i \in S: v_i = |V| - |S|$, $i \in \overline{S} v_i = -|S|$.

$\sum_i v_i = |S|(|V| - |S|) - |S|(|V| - |S|) = 0$

$\rightarrow v \perp \mathbf{1}$.

$v^T v = |S|(|V| - |S|)^2 + |S|^2(|V| - |S|) = |S|(|V| - |S|)(|V|)$.

$v^T M v = \frac{1}{d} \sum_{e=(i,j)} x_i x_j$.

Same side endpoints: like $v^T v$.

Different side endpoints: $-|S|(|V| - |S|)$

$v^T M v = v^T v - (2|E(S,S)||S|(|V| - |S|)$

$\frac{v^T M v}{v^T v} = 1 - \frac{2|E(S,\overline{S})|}{|S|}$

$\lambda_2 \geq 1 - 2h(S) \rightarrow h(G) \geq \frac{1-\lambda_2}{2}$

See you ...

Thursday.