Today.

Cuckoo hashing.

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Johnson-Lindenstrass.

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Hashing with two choices: max load $O(\log\log n)$.

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$O(1)$ time on average.

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Points: $x_1, \ldots, x_n \in \mathbb{R}^d$.

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"Projecting and scaling by $\sqrt{\frac{d}{k}}$ preserves all pairwise distances w/in factor of $1 \pm \varepsilon$."

Random subspace.

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remove projection onto previous subspace.

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Project x into subspace spanned by $v_1, v_2, \cdots, v_k$.

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y_i is ith coordinate of random vector z.

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k is large enough $\rightarrow$
$\approx (1 \pm \varepsilon)\sqrt{\frac{k}{d}}$ with decent probability.

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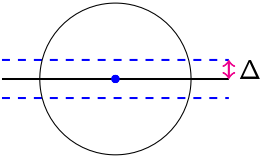

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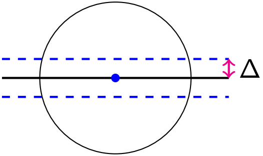

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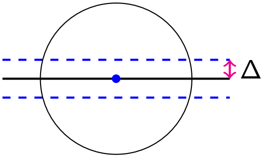
$z \geq \Delta$ from equator of sphere.



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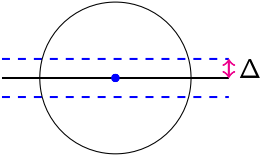

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$z \geq \Delta$ from equator of sphere.
Point on “Δ-spherical cap”.

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z is uniformly random unit vector.

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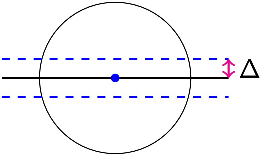

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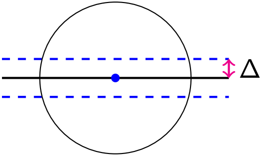

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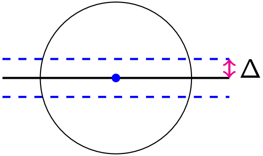

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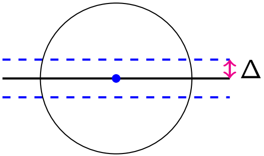

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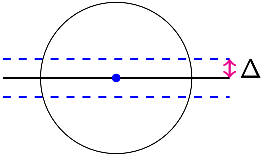

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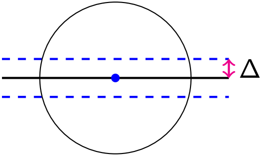

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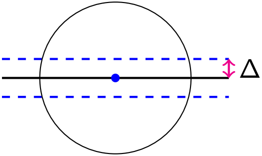

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Locality Preserving Hashing

Find nearby points in high dimensional space.

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  Use grid hash function.

Implementing Johnson-Lindenstraus

Random vectors

Implementing Johnson-Lindenstraus

Random vectors have many bits

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Use random bit vectors: $\{-1, +1\}^d$ instead.

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Coordinate for bit vector b.

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