Today.

Cuckoo hashing.

Johnson-Lindenstrass.

Cuckoo hashing.

Hashing with two choices: max load $O(\log\log n)$.

Cuckoo hashing:
Array. Two hash functions h_1, h_2.

Insert x: place in $h_1(x)$ or $h_2(x)$ if space.
Else bump elt y in $h_i(x)$ u.a.r. for $i \in [1,2]$.
Bump y,x: place y in $h_j(y)$ where $j \neq i$ if space.
Else bump y' in $h_j(y)$. And so on.

If go too long. Fail. Rehash entire hash table.
Fails if cycle for insert.

C_ℓ - event of cycle of length ℓ at a vertex.

$$\Pr[C_\ell] \leq \binom{m}{\ell}\binom{n}{\ell}\left(\frac{\ell}{n}\right)^{2(\ell)} \leq \left(\frac{e^2}{8}\right)^{\ell} \qquad (1)$$

Probability that an insert hits a cycle of length $\ell \leq \frac{\ell}{n}\left(\frac{e^2}{8}\right)^{\ell}$

Rehash every $\Omega(n)$ inserts (if $\leq n/8$ items in table.)
$O(1)$ time on average.

Johnson-Lindenstrass

Points: $x_1,\ldots,x_n \in \mathbb{R}^d$.

Random $k = \frac{c\log n}{\varepsilon^2}$ dimensional subspace.

Claim: with probability $1 - \frac{1}{n^{c-2}}$,

$$(1-\varepsilon)\sqrt{\frac{k}{d}}|x_i - x_j| \leq |y_i - y_j| \leq (1+\varepsilon)\sqrt{\frac{k}{d}}|x_i - x_j|$$

"Projecting and scaling by $\sqrt{\frac{d}{k}}$ preserves all pairwise distances w/in factor of $1 \pm \varepsilon$."

Random subspace.

Method 1:
Pick unit v_1 ,
v_2 orthogonal to v_1,
...
v_k orthogonal to previous vectors...

Method 2:
Choose k vectors $v_1,\ldots,v_k$
Gram Schmidt orthonormalization of $k \times d$ matrix where rows are v_i.
remove projection onto previous subspace.

Projections.

Project x into subspace spanned by $v_1, v_2, \cdots, v_k$.

$y_1 = x\cdot v_1, y_2 = x\cdot, v_2, \cdots, y_k = x\cdot v_k$

Projection: $(y_1,\ldots,y_k)$.

Have: Arbitrary vector, random k-dimensional subspace.

View As: Random vector, standard basis for k dimensions.

Orthogonal U - rotates $v_1,\ldots,v_k$ onto $e_1,\ldots,e_k$

$y_i = \langle v_i|x\rangle = \langle Uv_i|Ux\rangle = \langle e_i|Ux\rangle = \langle e_i|z\rangle$

Inverse of U maps e_i to random vector v_i

$z = Ux$ is uniformly distributed on d sphere for unit $x \in \mathbb{R}^d$.

y_i is ith coordinate of random vector z.

Expected value of y_i.

Random projection: first k coordinates of random unit vector, z_i.

$E[\sum_{i\in[d]} z_i^2] = 1$. Linearity of Expectation.

By symmetry, each z_i is identically distributed.

$E[\sum_{i\in[k]} z_i^2] = \frac{k}{d}$. Linearity of Expectation.

Expected length is $\sqrt{\frac{k}{d}}$.

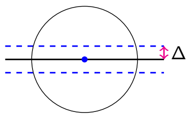
Johnson-Lindenstrass: close to expectation.
k is large enough $\rightarrow$
$\approx (1\pm\varepsilon)\sqrt{\frac{k}{d}}$ with decent probability.

Concentration Bounds.

z is uniformly random unit vector.
Random point on the unit sphere. $E[\sum_{i \in [k]} z_i^2] = \frac{k}{d}$.

Claim: $\Pr[|z_1| > \frac{t}{\sqrt{d}}] \leq e^{-t^2/2}$

Sphere view: surface "far" from equator defined by e_1.



$|z_1| \geq \Delta$ if
$z \geq \Delta$ from equator of sphere.
Point on "Δ-spherical cap".

Area of caps
$\leq$ S.A. of sphere of radius $\sqrt{1-\Delta^2}$
$\propto r^d = (1-\Delta^2)^{d/2}$
$\propto \left(1 - \frac{t^2}{d}\right)^{d/2} \approx e^{\frac{-t^2}{2}}$
Constant of $\propto$ is unit sphere area. □

$\Pr[\text{any } z_i^2 > (2 \log d) E[z_i^2]]$ is small.

Many coordinates.

Argued $\Pr[\text{any } z_i^2 > (2\log d)E[z_i^2]]$ is small.
Total Length? $z = \sqrt{z_1^2 + z_2^2 + \cdots z_k^2}$.

$\Pr[\left|\sqrt{(z_1^2 + z_2^2 + \cdots + z_k^2)} - \sqrt{\frac{k}{d}}\right| > t] \leq e^{-t^2 d/2}$

Substituting $t = \varepsilon\sqrt{\frac{k}{d}}$, $k = \frac{c \log n}{\varepsilon^2}$.

$\Pr[\left|\sqrt{z_1^2 + z_2^2 + \cdots + z_k^2} - \sqrt{\frac{k}{d}}\right| > \varepsilon\sqrt{\frac{k}{d}}] \leq e^{-\varepsilon^2 k} = e^{-c \log n} = \frac{1}{n^c}$

Johnson-Lindenstraus: For n points, $x_1, \ldots, x_n$, all distances preserved to within $1 \pm \varepsilon$ under $\sqrt{\frac{k}{d}}$-scaled projection above.

View one pair $x_i - x_j$ as vector.
Scale to unit.
Projection fails to preserve $|x_i - x_j|$
with probability $\leq \frac{1}{n^c}$
Scaled vector length also preserved.

$\leq n^2$ pairs plus union bound
$\rightarrow$ prob any pair fails to be preserved with $\leq \frac{1}{n^{c-2}}$.

Locality Preserving Hashing

Find nearby points in high dimensional space.
Points could be images!

Hash function $h(\cdot)$ s.t. $h(x_i) = h(x_j)$ if $d(x_i, x_j) \leq \delta$.

Low dimensions: grid cells give $\sqrt{d}$-approximation.
Not quite a solution. Why?
Close to grid boundary.
Find close points to x:
Check grid cell and neighboring grid cells.

Project high dimensional points into low dimensions.

Use grid hash function.

Implementing Johnson-Lindenstraus

Random vectors have many bits

Use random bit vectors: $\{-1, +1\}^d$ instead.

Almost orthogonal.

Project z.

Coordinate for bit vector b.
$C_l = \frac{1}{\sqrt{d}} \sum_i b_i z_i$

$E[C_l^2] = E[\frac{1}{d}\sum_{i,j} b_i b_j z_i z_j] = \frac{1}{d}\sum_{i,j} E[b_i b_j] z_i z_j = \frac{1}{d}\sum_i z_i^2 = \frac{1}{d}$

$E[\sum_l C_l^2] = \frac{k}{d}$

Binary Johnson-Lindenstrass

Project onto $[-1, +1]$ vectors.
$E[C] = E[\sum_l C_l^2] = \frac{k}{d}$

Concentration?

$$\Pr\left[|C - \frac{k}{d}| \geq \varepsilon\frac{k}{d}\right] \leq e^{-\varepsilon^2 k}$$

Choose $k = \frac{c \log n}{\varepsilon^2}$.
$\rightarrow$ failure probability $\leq 1/n^c$.

Analysis Idea.

$$\Pr\left[|C - \frac{k}{d}| \geq \varepsilon\frac{k}{d}\right] \leq e^{-\varepsilon^2 k}$$

Variance of C^2? Recall $C_l = \frac{1}{\sqrt{d}} \sum_i b_i z_i$

$Var(C) \leq \left(\frac{k}{d^2}\right)(\sum_i z_i{}^4 + 4\sum_{i,j} z_i^2 z_j^2) \leq \left(\frac{k}{d^2}\right) 2(\sum_i z_i^2)^2 \leq \frac{2k}{d^2}$.

Roughly normal (gaussian):
Density $\propto e^{-t^2/2}$ for t std deviations away.

So, assuming normality
$\sigma = \frac{\sqrt{2k}}{d}$, $t = \frac{\varepsilon\frac{k}{d}}{\frac{\sqrt{2k}}{d}} = \varepsilon\sqrt{k}/\sqrt{2}$.

Probability of failure roughly $\leq e^{-t^2/2}$
$\rightarrow e^{\varepsilon^2 k/4}$

"Roughly normal." Chernoff, Berry-Esseen, Central Limit Theorems.

Summary

Cuckoo hashing.

Two hash functions. Few cycles in random sparse graph.
Chaining works!

Johnson-Lindenstrass.
$O(\log n)$ dimensions give good approximation of distances.