Today

Gradient Descent.

Minimizing a convex function.

$$\min f(x)$$

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One dimension: go left or right with a magnitude.

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Depends on function.

Assume $\|\nabla f(x) - \nabla f(y)\| \leq L\|(x - y)\|$ for any x, y.

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Intuitively, a bound on the second derivative.

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Idea: $\nabla f(x_t)$.

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Idea: $\nabla f(x_t)$.

$|\nabla f(x_{t+1}) - \nabla f(x_t)| \leq L\|x_{t+1} - x_t\|^2$

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$|\nabla f(x_{t+1}) - \nabla f(x_t)| \leq L\|x_{t+1} - x_t\|^2 \leq L(\frac{1}{2L})\nabla f(x_{t+1})$

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$|\nabla f(x_{t+1}) - \nabla f(x_t)| \leq L\|x_{t+1} - x_t\|^2 \leq L(\frac{1}{2L})\nabla f(x_{t+1}) \leq \frac{1}{2}\nabla f(x_{t+1})$.

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For $\nabla f(y) > \nabla f(x_t)/2$, for all $y \in [x_t, x_{t+1}]$.

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For $\nabla f(y) > \nabla f(x_t)/2$, for all $y \in [x_t, x_{t+1}]$.

Thus, the function decreases by $\|\nabla f(x_t)\|^2/4L$.

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$\|\nabla f(x)\| \geq \frac{f(x)-f(x*)}{\|x-x*\|}$

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$$\|\nabla f(x)\| \geq \frac{f(x) - f(x*)}{\|x - x*\|} \implies f(x_{t+1}) \leq f(x_t) - \frac{1}{4L}\left(\frac{f(x_t) - f(x*)}{\|x_t - x^*\|}\right)^2$$

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Let $|f(x_t) - f(x*)| \in [\Delta, 2\Delta]$ for $t \in [0, k]$

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$\implies$ Error halves in k iterations.

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Geometric in Δ, so for arbitrary ε: $k = O((\frac{L\|x_t-x^*\|^2}{\varepsilon})$.

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$$\begin{aligned} f(x_{t+1}) &\leq f(x^*) + \nabla f(x_t)(x_t - x^*) - \frac{1}{2L}\|\nabla f(x_t)\|^2 \\ &\leq f(x^*) + \frac{1}{\alpha}(x_t - x^*(x - x^*) - \frac{1}{2L}\|\nabla f(x_t)\|^2 \end{aligned}$$

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Sum over iterations.

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$\sum_{t=1}^{k}(f(x_t) - f(x^*)) \leq \frac{L}{2}(\|x_0 - x^*\|^2 - \|x_k - x^*\|^2)$

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Since $f(x_t)$ is nonincreasing.

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$$f(x_k) - f(x^*) \leq \frac{1}{k}\sum_{t=1}^{k}(f(x_t) - f(x^*) \leq \frac{L\|x_0 - x^*\|^2}{2k}.$$

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Strong (strictly) Convexity: $f(x) - m\|x\|^2$ is convex for some $m > 0$.

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$\nabla^2 f(x) \succeq mI$ for all x.

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$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Strictly convex with $m = 2$.

Strong Convexity

Strong (strictly) Convexity: $f(x) - m\|x\|^2$ is convex for some $m > 0$.

$f(x) = 5x$? $f(x) = 5x^2$? $f(x) = 5x^3$?

$f(x, y) = x^2 + y^2$? $f(x, y) = 5x + 6y$?

If $f(x)$ is twice differentiable.

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Hessian for $f(x, y) = x^2 + xy + y^2$.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Strictly convex with $m = 1$.

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$c = (1 - O(m/L))$.

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So decreases by $(1 - \Theta(\frac{m^2}{L^2}))$ in each step.

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Better analysis: $(1 - \Theta(m/L))$ fraction in each step.

Next time.

Accelerated Gradient Descent.