Lagrange Multipliers.

Lagrangian Dual.

Find x, subject to

$f_i(x) \leq 0, i = 1, \ldots m.$

Remember calculus (constrained optimization.)

Lagrangian: $L(x,\lambda) = \sum_{i=1}^{m} \lambda_i f_i(x)$

$\lambda_i \geq 0$ - Lagrangian multiplier for inequality i.

For feasible solution x, $L(x,\lambda)$ is

(A) non-negative in expectation

(B) positive for any λ.

(C) non-positive for any valid λ.

If $\exists \lambda \geq 0$, where $L(x,\lambda)$ is positive for all x

(A) there is no feasible x.

(B) there is no x, λ with $L(x,\lambda) < 0$.

Lagrangian:constrained optimization.

$$\min \quad f(x)$$
$$\text{subject to } f_i(x) \leq 0, \qquad i = 1, \ldots, m$$

Lagrangian function:

$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$

If (primal) x has value v $f(x) = v$ and all $f_i(x) \leq 0$
For all $\lambda \geq 0$ have $L(x,\lambda) \leq v$
Maximizing λ, only positive λ_i when $f_i(x) = 0$
which implies $L(x,\lambda) \geq f(x) = v$

If there is λ with $L(x,\lambda) \geq \alpha$ for all x
Optimum value of program is at least α

Primal problem:
x, that minimizes $L(x,\lambda)$ over all $\lambda \geq 0$.

Dual problem:
λ, that maximizes $L(x,\lambda)$ over all x.

Why important: KKT.

Karash, Kuhn and Tucker Conditions.

$$\min \quad f(x)$$
$$\text{subject to } f_i(x) \leq 0, \qquad i = 1, \ldots, m$$

$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$

Local minima for feasible x^*.

There exist multipliers λ, where

$\nabla f(x^*) + \sum_i \lambda_i \nabla f_i(x^*) = 0$

Feasible primal, $f_i(x^*) \leq 0$, and feasible dual $\lambda_i \geq 0$.

Complementary slackness: $\lambda_i f_i(x^*) = 0$.

Launched nonlinear programming! See paper.

Linear Program.

$\min cx, Ax \geq b$

$$\min \quad c \cdot x$$
$$\text{subject to } b_i - a_i \cdot x \leq 0, \qquad i = 1, \ldots, m$$

Lagrangian (Dual):

$L(\lambda, x) = cx + \sum_i \lambda_i (b_i - a_i x)$.

or

$L(\lambda, x) = -(\sum_j x_j (a_j \lambda - c_j)) + b\lambda$.

Best λ?
$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

$\max b\lambda, \lambda^T A = c, \lambda \geq 0$
Duals!

Linear Systems...

Linear Systems

$Ax = b$

Find x.

Gaussian elimination: $O(n^3)$
$O(n^{2.36\cdots})$ with fast matrix multiplication.

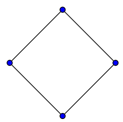
Iterative Methods: $O(nm \log \frac{1}{\varepsilon})$ to ε approximate.
For today: where m is sum of nonzeros in matrix.
For positive semidefinite matrix.

Today: $\tilde{O}(m)$ for Laplacian matrices.
Laplacian: $dI - A$ where A is adjacency matrix of a graph.
$\rightarrow$ symmetric diagonally dominant matrices by reduction.

Electrical Flow: a detour.

A graph $G = (V, E)$.
Circuit: nodes V, resistors E, value 1 (for today.)

Given $\chi : V \rightarrow \Re$
Find flow that routes χ and minimizes
$\sum_e f(e)^2$.



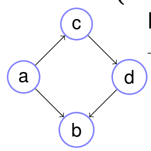
Claim: Minimizer is electrical flow.

Flow corresponds to flow induced by a set of potentials.

Some Matrices.

Given $G = (V, E)$, arbitrarily orient edges.

$$B_{v,e} = \begin{cases} -1 & e = (u,v) \\ 1 & e = (v,u) \\ 0 & \text{otherwise} \end{cases} \qquad L_{u,v} = \begin{cases} d & u = v \\ -1 & (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$



B

	a	b	c	d
(a,b)	1	-1	0	0
(a,c)	1	0	-1	0
(c,d)	0	0	1	-1
(d,b)	0	-1	0	1

L

	a	b	c	d
a	2	-1	-1	0
b	-1	2	0	-1
c	-1	0	2	-1
d	0	-1	-1	2

Fun facts: $\mathbf{f} \in \Re^{|E|}$
$[B^T f]_u = \sum_{e=(u,v)} f_e - \sum_{e=(v,u)} f_e$
$B^T B = L$
$[Bx]_{e=(u,v)} = x_u - x_v$
$x^T L x = \sum_{e=(u,v)} (x_u - x_v)^2$

Duality..

Given G, χ, $\chi \perp 1$

Minimize $|f|^2$ subject to $B^T f = \chi$.

Lagrangian: $L(\phi, f) = \sum_e f(e)^2 + 2\phi^T(\chi - B^T f)$

Lagrangian Dual: Find ϕ that maximizes $\min_f L(\phi, f)$.

Given ϕ, minimize $L(\phi, f)$? Calculus.

For $e = (u,v)$
$2f(e) + 2(\phi_v - \phi_u) = 0$ (Minimum when partial derivatives = 0.)

$\rightarrow f(e) = (\phi_u - \phi_v)$ Potential differences!!!

Matrix Form: $f = B\phi$ Again, flows should be potential differences.

Dual problem: Find ϕ that maximizes ...
$\max_\phi 2\phi^T \chi - \phi^T L \phi$
Note: want $\phi^T L \phi = \sum_e (\phi_u - \phi_v)^2$ to be small.
Minimize Squared Potential differences!

Why did we take dual?

Dual problem:
Find ϕ that maximizes ...
$\max_\phi 2\phi\chi - \phi L \phi$

Take the derivative:
$L\phi - \chi$
$L\phi = \chi$ at optimal point!

Optimal potential is solution to a Laplacian linear system.

Also useful for convergence.

Algorithm maintains feasible ϕ, f,

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi^T L \phi$

Duality gap is "distance" from optimal!

Algorithm: Work on flow and potentials.
To drive gap to 0.

Alg.

Given: χ, G

Take a spanning tree T of G. (Which tree?)

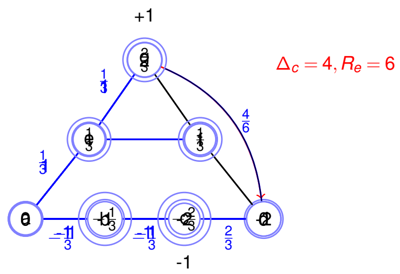
Route flow, f, to satisfy χ through T
Compute, ϕ, using tree ; $\phi_s = 0$, add f_e through T

Repeat:
Choose non-tree edge $e = (u,v)$ (Which non-tree edge?)
$f(e) = (\phi_u - \phi_v)/(l_T(u,v) + 1)$
($l_T(u,v)$ path length in T)
Route excess on path through tree.

Which Tree?
Claim: Linear time algorithm for T w/ stretch $O(m \log n \log \log n)$!
Stretch: $\sum_{e=(u,v)} l_T(u,v)$

Which non-tree edge?
Choose an edge w/prob. proportional to $l_T(e)$.

Finds $(1+\varepsilon)$ approximation in $O(m \log n \log \log n \log(\frac{n}{\varepsilon}))$!



Energy reduction.

Given T,$e=(u,v)$, let $R_e = l_T(u,v)+1$.

Algorithm:

Repeatedly "Fix" edge $e=(u,v)$.

Route $-\delta = -\frac{\sum_{e'\in C_e} f(e')}{R_e}$ flow around cycle induced in T: C_e

(assume e' are oriented around cycle.)

Difference in energy from f and f'.

$\sum_{e'\in C_e}(f(e')-\delta)^2-(f(e'))^2 = \sum_{e'\in C_e} -2f(e')\delta+\delta^2$
$= -(2\delta\sum_{e'\in C_e} f(e'))+R_E\delta^2$

Note: $\sum_{e'\in C_e} f(e') = R_e\delta$

$\rightarrow -\Delta^2_{C_e}/R_e$ where $\Delta_{C_e} = \sum_{e'\in C_e} f_(e')$.

Fix a part of the potential difference, Δ_{c_e} around cyle!!

$\rightarrow$ reduction of $\Delta^2_{C_e}/R_e$ in energy!

Fix $1/R_e$ of a cycle violation!

Duality Gap?

Algorithm maintains feasible ϕ, f, $(B^T f = \chi)$

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi L\phi$

ϕ is tree induced voltages.

Total Duality Gap?

Gap: $|f|^2-(2\phi^T\chi - \phi^T L\phi)$.

$= |f|^2 - 2\phi^T B^T f + \phi^T B^T B\phi$ where $B^T f = \chi$ and $L = B^T B$.
$= (f-B\phi)^T(f-B\phi)$.

Gap $= \sum_e (f(e)-\Delta_\phi(e))^2$ Difference between ϕ flow and f.

$\Delta_\phi(u,v) = \sum_{e\in P_{u,v}} -f(e)$. assume $f(e)$ is oriented around cycle.

For $e\in T$, $\Delta_\phi(e)=0$. For $e\notin T$ $f(e)+\sum_{e\in P_e} f(e) = \Delta_{c_e}(f)$

Duality Gap: $\sum_{e\notin T}\sum_e \Delta_{C_e}(f)^2$

Total distance from optimal is cycle violations!

Claim: E[change in energy$|Gap$] $= \frac{Gap}{\tau}$ (τ is stretch of E in T.)

Duality Gap: $\sum_{e\notin T}\Delta_{C_e}(f)^2$

Choose edge e reduce energy by $-\frac{\Delta^2_{C_e}}{R_e}$.

Choose edge with probability $\frac{R_e}{\tau}$.

Expected reduction $-\sum_e \frac{R_e}{\tau}\frac{\Delta^2_{C_e}}{R_e} = -\sum_e \frac{\Delta^2_{C_e}}{\tau}$ □

Duality Gap reduces by $(1-1/\tau)$ every iteration on expectation.

$O(\tau\log(n/\varepsilon))$ iterations gives $(1+\varepsilon)$ approximation.

$\tau = O(m\log n\log\log n)$ ...

$\tilde{O}(m)$ iterations

Iteration in $O(\log^2 n)$ time using balanced binary trees.

$\rightarrow \tilde{O}(m)$ time! ! ! ! ! ! ! ! ! ! ! !

See you ...

Tuesday.