Crazy Picture.
Maximum matching and simplex.

\[ \text{Blue constraints intersect.} \]
\[ \text{Blue constraints redundant.} \]
\[ \text{Blue constraints tight.} \]

\[ \text{Sum: } x + 2z + y. \]
Maximum matching and simplex.

\[
\begin{align*}
\text{Maximize} & \quad x + y + z \\
\text{subject to} & \quad x \leq 1 \\
& \quad x + z \leq 1 \\
& \quad z + y \leq 1 \\
& \quad y \leq 1 \\
& \quad x \geq 0 \\
& \quad y \geq 0 \\
& \quad z \geq 0
\end{align*}
\]

Blue constraints intersect.
Blue constraints redundant.
Blue constraints tight.

Sum:
\[x + 2z + y\]
Maximize $x + y + z$

- $x \leq 1$
- $x + z \leq 1$
- $z + y \leq 1$
- $y \leq 1$
- $x \geq 0$
- $y \geq 0$
- $z \geq 0$

Blue constraints intersect.
Maximum matching and simplex.

\[
\begin{align*}
\text{max } x + y + z \\
x &\leq 1 \\
x + z &\leq 1 \\
z + y &\leq 1 \\
y &\leq 1 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0
\end{align*}
\]

Blue constraints intersect.
Maximum matching and simplex.

\[
\begin{align*}
\text{max } & \quad x + y + z \\
\text{subject to } & \quad x \leq 1 \\
& \quad x + z \leq 1 \\
& \quad z + y \leq 1 \\
& \quad y \leq 1 \\
& \quad x \geq 0 \\
& \quad y \geq 0 \\
& \quad z \geq 0
\end{align*}
\]

Blue constraints intersect.
Maximum matching and simplex.

\[ \text{max } x + y + z \]
\[ x \leq 1 \]
\[ x + z \leq 1 \]
\[ z + y \leq 1 \]
\[ y \leq 1 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]

Blue constraints intersect.
Maximum matching and simplex.

\[
\begin{align*}
\text{max} & \quad x + y + z \\
& \quad x \leq 1 \\
& \quad x + z \leq 1 \\
& \quad z + y \leq 1 \\
& \quad y \leq 1 \\
& \quad x \geq 0 \\
& \quad y \geq 0 \\
& \quad z \geq 0 
\end{align*}
\]

Blue constraints intersect.
Maximum matching and simplex.

\[
\begin{align*}
\text{max} & \ x + y + z \\
\text{s.t.} & \ x \leq 1 \\
& \ x + z \leq 1 \\
& \ z + y \leq 1 \\
& \ y \leq 1 \\
& \ x \geq 0 \\
& \ y \geq 0 \\
& \ z \geq 0
\end{align*}
\]

Blue constraints intersect.
Maximum matching and simplex.

\[
\begin{align*}
\text{max } x + y + z \\
x &\leq 1 \\
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z + y &\leq 1 \\
y &\leq 1 \\
x &\geq 0 \\
y &\geq 0 \\
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z &\geq 0
\end{align*}
\]
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[ x + z \leq 1 \]
\[ z + y \leq 1 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[ x + z \leq 1 \]
\[ z + y \leq 1 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]

Blue constraints intersect.
Blue constraints redundant.
Blue constraints tight.

\[ \text{Augmenting Path.} \]
Via Gaussian Elimination!
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[ x + z \leq 1 \]
\[ z + y \leq 1 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]

Blue constraints tight.
Maximum matching and simplex.

\[
\text{max } x + y + z
\]

\[
x + z \leq 1
\]
\[
z + y \leq 1
\]
\[
x \geq 0
\]
\[
y \geq 0
\]
\[
z \geq 0
\]

Blue constraints tight.
Maximum matching and simplex.

\[
\begin{align*}
\text{max } & x + y + z \\
\text{s.t. } & x + z \leq 1 \\
& z + y \leq 1 \\
& x \geq 0 \\
& y \geq 0 \\
& z \geq 0
\end{align*}
\]

Blue constraints tight.

Augmenting Path.

Via Gaussian Elimination!
Maximum matching and simplex.

\[
\text{max } x + y + z \\
\]

\[
x + z \leq 1 \\
z + y \leq 1 \\
x \geq 0 \\
y \geq 0 \\
z \geq 0
\]

Blue constraints tight.
Maximum matching and simplex.

max \( x + y + z \)

\begin{align*}
  x + z & \leq 1 \\
  z + y & \leq 1 \\
  x & \geq 0 \\
  y & \geq 0 \\
  z & \geq 0
\end{align*}

Blue constraints tight.
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[ x + z \leq 1 \]
\[ z + y \leq 1 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]

Blue constraints tight.

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Blue constraints tight.
Maximum matching and simplex.

\[
\begin{align*}
\max & \quad x + y + z \\
\text{s.t.} & \quad x + z \leq 1 \\
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& \quad x \geq 0 \\
& \quad y \geq 0 \\
& \quad z \geq 0 \\
\end{align*}
\]

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Maximum matching and simplex.

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\[ x + z \leq 1 \]
\[ z + y \leq 1 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]

Blue constraints tight.
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[ x + z \leq 1 \]
\[ z + y \leq 1 \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]

Blue constraints tight.

Augmenting Path.
Maximum matching and simplex.

max \( x + y + z \)

\[
\begin{align*}
x + z & \leq 1 \\
z + y & \leq 1
\end{align*}
\]

\[
\begin{align*}
x & \geq 0 \\
y & \geq 0 \\
z & \geq 0
\end{align*}
\]

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[ x + z \leq 1 \]
\[ z + y \leq 1 \]

\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \]

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

\[
\begin{align*}
\text{max } & x + y + z \\
x + z & \leq 1 \\
z + y & \leq 1 \\
x & \geq 0 \\
y & \geq 0 \\
z & \geq 0
\end{align*}
\]

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

Maximize: \( x + y + z \)

Subject to:
- \( x + z \leq 1 \)
- \( z + y \leq 1 \)
- \( x \geq 0 \)
- \( y \geq 0 \)
- \( z \geq 0 \)

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[ x + z \leq 1 \quad a \]
\[ z + y \leq 1 \quad b \]
\[ x \geq 0 \]
\[ y \geq 0 \]
\[ z \geq 0 \quad c \]

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

$$\text{max } x + y + z$$

$$x + z \leq 1 \quad a = 1$$
$$z + y \leq 1 \quad b = 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0 \quad c = 1$$

Blue constraints tight.

Sum: $x + 2z + y$.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

$$\text{max } x + y + z$$

$$x + z \leq 1 \quad a = 1$$
$$z + y \leq 1 \quad b = 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0 \quad c = 1$$

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[
\begin{align*}
  x + z & \leq 1 \quad a = 1 \\
  z + y & \leq 1 \quad b = 1 \\
  x & \geq 0 \\
  y & \geq 0 \\
  z & \geq 0 \quad c = 1
\end{align*}
\]

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Maximum matching and simplex.

\[ \text{max } x + y + z \]

\[
\begin{align*}
    x + z & \leq 1 \quad a = 1 \\
    z + y & \leq 1 \quad b = 1 \\
    x & \geq 0 \\
    y & \geq 0 \\
    z & \geq 0 \quad c = 1
\end{align*}
\]

Blue constraints tight.

Augmenting Path. Via Gaussian Elimination!
Convex Separator.

Farkas

Strong Duality!!!!!
Convex Separator.
Convex Separator.

Farkas
Convex Separator.

Farkas

Strong Duality!!!!!
Convex Separator.
Farkas
Strong Duality!!!!! Maybe.
Linear Equations.

\[ Ax = b \]
Linear Equations.

\[ Ax = b \]

\( A \) is \( n \times n \) matrix...
Linear Equations.

\[ Ax = b \]

\( A \) is \( n \times n \) matrix...

..has a solution.
Linear Equations.

\[ Ax = b \]

\[ A \text{ is } n \times n \text{ matrix...} \]

..has a solution.

If rows of \( A \) are linearly independent.

\[ y^T A \neq 0 \text{ for any } y \]

..or if \( b \) in subspace of \( A \).

\[ x_1 \quad x_2 \quad x_3 \quad \text{ok} \quad b \quad \text{bad} \]
Linear Equations.

\[ Ax = b \]

\( A \) is \( n \times n \) matrix...

..has a solution.

If rows of \( A \) are linearly independent.  
\[ y^T A \neq 0 \text{ for any } y \]
Linear Equations.

\[ Ax = b \]

\( A \) is \( n \times n \) matrix...

..has a solution.

If rows of \( A \) are linearly independent.

\[ y^T A \neq 0 \text{ for any } y \]

..or if \( b \) in subspace of \( A \).
Linear Equations.

\[ Ax = b \]

\( A \) is \( n \times n \) matrix...

..has a solution.

If rows of \( A \) are linearly independent.

\[ y^T A \neq 0 \text{ for any } y \]

..or if \( b \) in subspace of \( A \).
Strong Duality.
Strong Duality.

Later.
Strong Duality.

Strong Duality.

Later. Actually. No. Now
Strong Duality.

Later. Actually. No. Now ...ish.

Special Cases:
Strong Duality.

Later. Actually. No. Now ...ish.

Special Cases:
  min-max 2 person games and experts.
Later. Actually. No. Now ...ish.

Special Cases:
- min-max 2 person games and experts.
- Max weight matching and algorithm.
Later. Actually. No. Now ...ish.

**Special Cases:**
- min-max 2 person games and experts.
- Max weight matching and algorithm.
- Approximate: facility location primal dual.
Strong Duality.

Later. Actually. No. Now ...ish.
Special Cases:
  min-max 2 person games and experts.
  Max weight matching and algorithm.
  Approximate: facility location primal dual.

Today: Geometry!
For a convex body $P$ and a point $b$, $b \in P$ or hyperplane separates $P$ from $b$. 
Convex Body and point.

For a convex body $P$ and a point $b$, $b \in P$ or hyperplane separates $P$ from $b$.

$v, \alpha$, where $v \cdot x \leq \alpha$ and $v \cdot b > \alpha$. 
For a convex body $P$ and a point $b$, $b \in P$ or hyperplane separates $P$ from $b$.

$v, \alpha$, where $v \cdot x \leq \alpha$ and $v \cdot b > \alpha$. 
For a convex body $P$ and a point $b$, $b \in P$ or hyperplane separates $P$ from $b$.

$v, \alpha$, where $v \cdot x \leq \alpha$ and $v \cdot b > \alpha$.

Point $p$ where $(x - p)^T(b - p) < 0$.
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T(b - p) < 0$
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T(b - p) < 0$

Proof: Choose $p$ to be closest point to $b$ in $P$. 
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where 
\[(x - p)^T (b - p) < 0\]

**Proof:** Choose $p$ to be closest point to $b$ in $P$.

Done
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T(b - p) < 0$

**Proof:** Choose $p$ to be closest point to $b$ in $P$.

Done or $\exists\ x \in P$ with $(x - p)^T(b - p) \geq 0$
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T (b - p) < 0$

Proof: Choose $p$ to be closest point to $b$ in $P$.

Done or $\exists x \in P$ with $(x - p)^T (b - p) \geq 0$

$(x - p)^T (b - p) \geq 0$
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T(b - p) < 0$

Proof: Choose $p$ to be closest point to $b$ in $P$.

Done or $\exists x \in P$ with $(x - p)^T(b - p) \geq 0$

$(x - p)^T(b - p) \geq 0 \rightarrow \leq 90^\circ$ angle between $x - p$ and $b - p$. 
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T(b - p) < 0$

Proof: Choose $p$ to be closest point to $b$ in $P$.

Done or $\exists x \in P$ with $(x - p)^T(b - p) \geq 0$

$(x - p)^T(b - p) \geq 0$

$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x - p}$ and $\overrightarrow{b - p}$.

Must be closer point $b$ on line from $p$ to $x$. 
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T(b - p) < 0$

**Proof:** Choose $p$ to be closest point to $b$ in $P$.

Done or $\exists x \in P$ with $(x - p)^T(b - p) \geq 0$

$(x - p)^T(b - p) \geq 0$

$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x - p}$ and $\overrightarrow{b - p}$.

Must be closer point $b$ on line from $p$ to $x$.

All points on line to $x$ are in polytope.
Proof.

For a convex body $P$ and a point $b$, $b \in A$ or there is point $p$ where $(x - p)^T(b - p) < 0$

Proof: Choose $p$ to be closest point to $b$ in $P$.

Done or $\exists x \in P$ with $(x - p)^T(b - p) \geq 0$

$(x - p)^T(b - p) \geq 0$

$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x - p}$ and $\overrightarrow{b - p}$.

Must be closer point $b$ on line from $p$ to $x$.

All points on line to $x$ are in polytope.

Contradicts choice of $p$ as closest point to $b$ in polytope.
More formally.

Squared distance to $b$ from $p + (x - p)\mu$

$\theta$ is the angle between $x - p$ and $b - p$. 

$\ell = \mu |x - p|$
More formally.

Squared distance to $b$ from $p + (x - p)\mu$
point between $p$ and $x$
More formally.

Squared distance to $b$ from $p + (x - p)\mu$
point between $p$ and $x$

$$
(|p - b| - \mu|x - p|\cos\theta)^2 + (\mu|x - p|\sin\theta)^2
$$

More formally.

Squared distance to $b$ from $p + (x - p)\mu$
point between $p$ and $x$

$(|p - b| - \mu|x - p|\cos\theta)^2 + (\mu|x - p|\sin\theta)^2$

$\theta$ is the angle between $x - p$ and $b - p$. 
More formally.

Squared distance to $b$ from $p + (x - p)\mu$

point between $p$ and $x$

$(|p - b| - \mu|x - p|\cos\theta)^2 + (\mu|x - p|\sin\theta)^2$

$\theta$ is the angle between $x - p$ and $b - p$. 

Distance to new point.
More formally.

Squared distance to \( b \) from \( p + (x - p)\mu \)
point between \( p \) and \( x \)
\[
(\|p - b\| - \mu|x - p|\cos \theta)^2 + (\mu|x - p|\sin \theta)^2
\]
\( \theta \) is the angle between \( x - p \) and \( b - p \).

\[
\ell = \mu|x - p|
\]
\( p + \mu(x - p) \)

Distance to new point.

Simplify:
More formally.

Squared distance to $b$ from $p + (x - p)\mu$
point between $p$ and $x$

$(|p - b| - \mu |x - p| \cos \theta)^2 + (\mu |x - p| \sin \theta)^2$

$\theta$ is the angle between $x - p$ and $b - p$.

Distance to new point.

Simplify:

$|p - b|^2 - 2\mu |p - b||x - p| \cos \theta + (\mu |x - p|)^2$. 
More formally.

Squared distance to $b$ from $p + (x - p)\mu$
point between $p$ and $x$
$$(|p - b| - \mu|x - p|\cos \theta)^2 + (\mu|x - p|\sin \theta)^2$$
$\theta$ is the angle between $x - p$ and $b - p$.

Simplify:
$|p - b|^2 - 2\mu|p - b||x - p|\cos \theta + (\mu|x - p|)^2$.
Derivative with respect to $\mu$ ...
More formally.

Squared distance to $b$ from $p + (x - p)\mu$
point between $p$ and $x$

$$\left( |p - b| - \mu |x - p| \cos \theta \right)^2 + \left( \mu |x - p| \sin \theta \right)^2$$

$\theta$ is the angle between $x - p$ and $b - p$.

Distance to new point.

\[ \ell = \mu |x - p| \]

Simplify:

\[ |p - b|^2 - 2\mu |p - b| |x - p| \cos \theta + (\mu |x - p|)^2. \]

Derivative with respect to $\mu$ ...

\[ -2 |p - b| |x - p| \cos \theta + 2(\mu |x - p|^2). \]
More formally.

Squared distance to $b$ from $p + (x - p)\mu$
point between $p$ and $x$
\[(|p - b| - \mu |x - p| \cos \theta)^2 + (\mu |x - p| \sin \theta)^2\]
$\theta$ is the angle between $x - p$ and $b - p$.

Simplify:
\[|p - b|^2 - 2\mu |p - b| |x - p| \cos \theta + (\mu |x - p|)^2.\]
Derivative with respect to $\mu$ ...
\[-2|p - b| |x - p| \cos \theta + 2(\mu |x - p|^2)\]
which is negative for a small enough value of $\mu$
More formally.

Squared distance to \( b \) from \( p + (x - p)\mu \) point between \( p \) and \( x \):

\[
(|p - b| - \mu|x - p|\cos\theta)^2 + (\mu|x - p|\sin\theta)^2
\]

\( \theta \) is the angle between \( x - p \) and \( b - p \).

Distance to new point.

\[
l = \mu|x - p|
\]

\[
|p - b| - l\cos\theta
\]

Simplify:

\[
|p - b|^2 - 2\mu |p - b||x - p|\cos\theta + (\mu|x - p|)^2.
\]

Derivative with respect to \( \mu \) ...

\[
-2|p - b||x - p|\cos\theta + 2(\mu|x - p|^2).
\]

which is negative for a small enough value of \( \mu \) (for positive \( \cos\theta \).)
Theorems of Alternatives.
Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.
Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$. 
Generalization: exercise.

Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$.
That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$. 
Generalization: exercise.

Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$. That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$. Space is image of $A$. 

Affine subspace is columnspan of $A$. $y$ is normal. $y$ in nullspace for column span. $y^T b \neq 0 \Rightarrow b$ not in column span.

There is a separating hyperplane between any two convex bodies. Idea: Let closest pair of points in two bodies define direction.
Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$. That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$. Space is image of $A$. Affine subspace is columnspace of $A$. 

There is a separating hyperplane between any two convex bodies. Idea: Let closest pair of points in two bodies define direction.
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Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$.
That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$.
Space is image of $A$. Affine subspace is columnspace of $A$.
$y$ is normal.
Generalization: exercise.

Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$. That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$.

Space is image of $A$. Affine subspace is columnspan of $A$. $y$ is normal. $y$ in nullspace for column span.
Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$.
That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$.
Space is image of $A$. Affine subspace is columnspan of $A$.
y is normal. $y$ in nullspace for column span.
$y^T b \neq 0 \implies b$ not in column span.
Generalization: exercise.

Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$. That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$.

Space is image of $A$. Affine subspace is columnspan of $A$. $y$ is normal. $y$ in nullspace for column span.

$y^T b \neq 0 \implies b$ not in column span.

There is a separating hyperplane between any two convex bodies.
Generalization: exercise.

Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$.
That is, find $y$ where $y^T A = 0$ and $y^T b \neq 0$.
Space is image of $A$. Affine subspace is columnspan of $A$.
$y$ is normal. $y$ in nullspace for column span.
$y^T b \neq 0 \implies b$ not in column span.

There is a separating hyperplane between any two convex bodies.
Idea: Let closest pair of points in two bodies define direction.
Ax = b, x ≥ 0

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
= 
\begin{bmatrix}
1
\end{bmatrix}
\]

Coordinates s = b − Ax.

x ≥ 0 where s = 0?
\[ Ax = b, \ x \geq 0 \]
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1
\end{bmatrix}
\]

Coordinates \( s = b - Ax \).
\( x \geq 0 \) where \( s = 0 \)?
Ax = b, x ≥ 0

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
\end{bmatrix}
\]

Coordinates \( s = b - Ax \).

\( x \geq 0 \) where \( s = 0 \)?
Ax = b, \quad x \geq 0

\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix} x = \begin{bmatrix}
1 \\
1
\end{bmatrix}

Coordinates \ s = b - Ax.
\quad x \geq 0 \text{ where } s = 0？
Ax = b, x ≥ 0

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
\end{bmatrix}
\]

Coordinates s = b − Ax.
x ≥ 0 where s = 0?
Ax = b, x ≥ 0

Coordinates s = b - Ax.

where s = 0?
Ax = b, x ≥ 0
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
−1 \\
−1
\end{bmatrix}
\]

Coordinates s = b − Ax.

x ≥ 0 where s = 0?

Farkas A: Solution for exactly one of:
1. Ax = b, x ≥ 0
2. y^T A ≥ 0, y^T b < 0.
\[ Ax = b, \ x \geq 0 \]
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
-1 \\
-1 \\
\end{bmatrix}
\]

Coordinates \( s = b - Ax \).

\( x \geq 0 \) where \( s = 0 \)?
$Ax = b, \ x \geq 0$

$$
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
x \\
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
-1 \\
\end{bmatrix}
$$

Coordinates $s = b - Ax$. $x \geq 0$ where $s = 0$?

$y$ where $y^T(b - Ax) < y^T(0) < 0$ for all $x \geq 0$
Ax = b, x ≥ 0

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}x = \begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]

Coordinates s = b - Ax.

x ≥ 0 where s = 0?

y where \(y^T(b - Ax) < y^T(0) < 0\) for all \(x ≥ 0\) → \(y^Tb < 0\) and \(y^TA ≥ 0\).
$Ax = b$, $x \geq 0$

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix} x = \begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]

Coordinates $s = b - Ax$.
$x \geq 0$ where $s = 0$?

$y$ where $y^T (b - Ax) < y^T (0) < 0$ for all $x \geq 0$ $\rightarrow$ $y^T b < 0$ and $y^T A \geq 0$.

**Farkas A:** Solution for exactly one of:
\( Ax = b, \ x \geq 0 \)

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix} x = \begin{bmatrix}
-1 \\
-1
\end{bmatrix}
\]

Coordinates \( s = b - Ax \).

\( x \geq 0 \) where \( s = 0 \)?

\( y \) where \( y^T(b - Ax) < y^T(0) < 0 \) for all \( x \geq 0 \) \( \rightarrow y^Tb < 0 \) and \( y^TA \geq 0 \).

**Farkas A:** Solution for exactly one of:

1. \( Ax = b, \ x \geq 0 \)
\[ Ax = b, \ x \geq 0 \]
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
-1 \\
\end{bmatrix}
\]

Coordinates \( s = b - Ax \).
\( x \geq 0 \) where \( s = 0 \)?

\[ y \text{ where } y^T(b - Ax) < y^T(0) < 0 \text{ for all } x \geq 0 \rightarrow y^Tb < 0 \text{ and } y^TA \geq 0. \]

**Farkas A:** Solution for exactly one of:

1. \( Ax = b, \ x \geq 0 \)
2. \( y^TA \geq 0, \ y^Tb < 0. \)
Farkas 2

**Farkas A:** Solution for exactly one of:

\[ Ax = b, \quad x \geq 0 \]
\[ y^T A \geq 0, \quad y^T b < 0. \]
Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
**Farkas A:** Solution for exactly one of:

1. $Ax = b, x \geq 0$
2. $y^T A \geq 0, y^T b < 0$. 
Farkas 2

Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
(2) $y^T A \geq 0, y^T b < 0$.

Farkas B: Solution for exactly one of:
Farkas 2

**Farkas A:** Solution for exactly one of:
1. $Ax = b, x \geq 0$
2. $y^T A \geq 0, y^T b < 0$.

**Farkas B:** Solution for exactly one of:
1. $Ax \leq b$
Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
(2) $y^T A \geq 0, y^T b < 0$.

Farkas B: Solution for exactly one of:
(1) $Ax \leq b$
(2) $y^T A = 0, y^T b < 0, y \geq 0$. 
Strong Duality

(From Goemans notes.)

Primal P \[ z^* = \min c^T x \]
\[ Ax = b \]
\[ x \geq 0 \]

Dual D \[ w^* = \max b^T y \]
\[ A^T y \leq c \]
Strong Duality

(From Goemans notes.)

Primal P \[ z^* = \min c^T x \]
\[ Ax = b \]
\[ x \geq 0 \]

Dual D \[ w^* = \max b^T y \]
\[ A^T y \leq c \]

Weak Duality: \( x, y \) - feasible P, D: \( x^T c \geq b^T y \).
Strong Duality

(From Goemans notes.)

Primal P \( z^* = \min c^T x \)
\[ Ax = b \]
\[ x \geq 0 \]

Dual D \( w^* = \max b^T y \)
\[ A^T y \leq c \]

Weak Duality: \( x, y \)- feasible P, D: \( x^T c \geq b^T y \).

\[
x^T c - b^T y = x^T c - x^T A^T y
= x^T (c - A^T y)
\geq 0
\]
Strong duality If $P$ or $D$ is feasible and bounded then $z^* = w^*$. 
Strong duality If $P$ or $D$ is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$. 

**Strong duality** If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim**: Exists a solution to dual of value at least $z^*$. 
Strong duality If P or D is feasible and bounded then \( z^* = w^* \).

Primal feasible, bounded, value \( z^* \).

Claim: Exists a solution to dual of value at least \( z^* \).

\[ \exists y, y^T A \leq c, b^T y \geq z^*. \]
**Strong duality** If $P$ or $D$ is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$$\exists y, y^T A \leq c, b^T y \geq z^*.$$  

Want $y$. 
**Strong duality** If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want $y$.

$$
\begin{pmatrix}
A^T \\
-b^T
\end{pmatrix}

y \leq

\begin{pmatrix}
c \\
-z^*
\end{pmatrix}.
$$
**Strong duality** If \( P \) or \( D \) is feasible and bounded then \( z^* = w^* \).

Primal feasible, bounded, value \( z^* \).

**Claim:** Exists a solution to dual of value at least \( z^* \).

\[ \exists y, y^T A \leq c, b^T y \geq z^* . \]

Want \( y \).

\[ \begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix} . \]

If none, then Farkas B says

\[ \exists x, \lambda \geq 0 . \]

\[ (A \quad -b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0 \]

\[ (c^T \quad -z^*) \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0 \]
Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

Claim: Exists a solution to dual of value at least $z^*$.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want $y$.

$$
\begin{pmatrix}
A^T \\
-b^T
\end{pmatrix} y \leq \begin{pmatrix} c \\
-z^*
\end{pmatrix}.
$$

If none, then Farkas B says

$\exists x, \lambda \geq 0.$

$$
(A \quad -b) \begin{pmatrix} x \\
\lambda
\end{pmatrix} = 0
$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$
**Strong duality** If $P$ or $D$ is feasible and bounded then $z^* = w^*$.  
Primal feasible, bounded, value $z^*$.  

**Claim:** Exists a solution to dual of value at least $z^*$.  
\[ \exists y, y^T A \leq c, b^T y \geq z^*. \]
Want $y$.  
\[ \begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}. \]
If none, then Farkas B says\[ \exists x, \lambda \geq 0. \]
\[ (A - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0 \]
\[ (c^T - z^*) \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0 \]
\[ \exists x, \lambda \text{ with } Ax - b\lambda = 0 \text{ and } c^t x - z^* \lambda < 0 \]
Case 1: $\lambda > 0$.  

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want $y$.

$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}$.

If none, then Farkas B says

$\exists x, \lambda \geq 0.
\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0$

$\begin{pmatrix} c^T & -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^T x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A\left(\frac{x}{\lambda}\right) = b$, $c^T \left(\frac{x}{\lambda}\right) < z^*$.
**Strong duality** If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want $y$.

$\left(\begin{array}{c} A^T \\ -b^T \end{array} \right) y \leq \left( \begin{array}{c} c \\ -z^* \end{array} \right)$.

If none, then Farkas B says

$\exists x, \lambda \geq 0$.

$\left( \begin{array}{c} A & -b \end{array} \right) \left( \begin{array}{c} x \\ \lambda \end{array} \right) = 0$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^T x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!
**Strong duality** If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$\exists y, y^TA \leq c, b^Ty \geq z^*$.

Want $y$.

\[
\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.
\]

If none, then Farkas B says

$\exists x, \lambda \geq 0$.

\[
(A - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0
\]

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^Tx - z^*\lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b, c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^Tx < 0$.
**Strong duality** If P or D is feasible and bounded then \( z^* = w^* \).

Primal feasible, bounded, value \( z^* \).

**Claim:** Exists a solution to dual of value at least \( z^* \).

\[
\exists y, y^T A \leq c, b^T y \geq z^*.
\]

Want \( y \).

\[
\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.
\]

If none, then Farkas B says

\[
\exists x, \lambda \geq 0.
\]

\[
\begin{pmatrix} A \\ -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0
\]

\[
\begin{pmatrix} c^T \\ -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0
\]

\( \exists x, \lambda \) with \( Ax - b\lambda = 0 \) and \( c^T x - z^* \lambda < 0 \)

Case 1: \( \lambda > 0 \). \( A \left( \frac{x}{\lambda} \right) = b, c^T \left( \frac{x}{\lambda} \right) < z^* \). Better Primal!!

Case 2: \( \lambda = 0 \). \( Ax = 0, c^T x < 0 \).

Feasible \( \tilde{x} \) for Primal.
**Strong duality** If \( P \) or \( D \) is feasible and bounded then \( z^* = w^* \).

Primal feasible, bounded, value \( z^* \).

**Claim:** Exists a solution to dual of value at least \( z^* \).

\( \exists y, y^T A \leq c, b^T y \geq z^* \).

Want \( y \).

\[
\begin{pmatrix}
A^T \\
-b^T
\end{pmatrix} y \leq 
\begin{pmatrix}
c \\
-z^*
\end{pmatrix}.
\]

If none, then Farkas B says

\( \exists x, \lambda \geq 0. \)

\[
(A - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0
\]

\( (c^T - z^*) \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0 \)

\( \exists x, \lambda \) with \( Ax - b\lambda = 0 \) and \( c^T x - z^* \lambda < 0 \)

**Case 1:** \( \lambda > 0. \) \( A(\frac{x}{\lambda}) = b, \quad c^T(\frac{x}{\lambda}) < z^* \). Better Primal!!

**Case 2:** \( \lambda = 0. \) \( Ax = 0, \quad c^T x < 0. \)

Feasible \( \tilde{x} \) for Primal.

(a) \( \tilde{x} + \mu x \geq 0 \) since \( \tilde{x}, x, \mu \geq 0. \)
**Strong duality** If $P$ or $D$ is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$$\exists y, y^T A \leq c, b^T y \geq z^*.$$  

Want $y$.

$$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.$$  

If none, then Farkas B says

$$\exists x, \lambda \geq 0.$$  

$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0$$

$$\exists x, \lambda \text{ with } Ax - b\lambda = 0 \text{ and } c^T x - z^* \lambda < 0$$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T (\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0$, $c^T x < 0$.

Feasible $\tilde{x}$ for Primal.

(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.

(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. 
**Strong duality** If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want $y$.

$$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.$$ 

If none, then Farkas B says

$\exists x, \lambda \geq 0$. 

$$(A \ -b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^T x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0$, $c^T x < 0$.

Feasible $\tilde{x}$ for Primal.

(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.

(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible
**Strong duality** If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want $y$.

$$
\begin{pmatrix}
A^T \\
-b^T
\end{pmatrix}
y \leq \begin{pmatrix}
c \\
-z^*
\end{pmatrix}.
$$

If none, then Farkas B says

$\exists x, \lambda \geq 0$.

$$(A - b) \begin{pmatrix}
x \\
\lambda
\end{pmatrix} = 0$$

$$(c^T - z^*) \begin{pmatrix}
x \\
\lambda
\end{pmatrix} < 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^T x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T (\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0$, $c^T x < 0$.

Feasible $\tilde{x}$ for Primal.

(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.

(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible

$c^T (\tilde{x} + \mu x) = x^T \tilde{x} + \mu c^T x \to -\infty$ as $\mu \to \infty$
**Strong duality** If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value $z^*$.

**Claim:** Exists a solution to dual of value at least $z^*$.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want $y$.

$$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.$$  

If none, then Farkas B says

$\exists x, \lambda \geq 0$.

$$(A - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0$$

$$(c^T - z^*) \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^T x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0$, $c^T x < 0$.

Feasible $\tilde{x}$ for Primal.

(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.

(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu Ax = b + \mu \cdot 0 = b$. Feasible

$c^T(\tilde{x} + \mu x) = x^T\tilde{x} + \mu c^T x \rightarrow -\infty$ as $\mu \rightarrow \infty$

Primal unbounded!
See you on Tuesday.