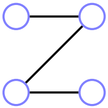
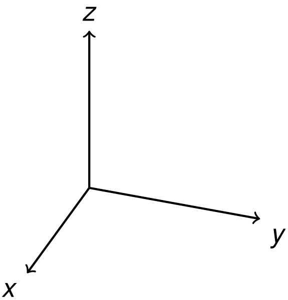
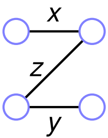
Crazy Picture.

Maximum matching and simplex.

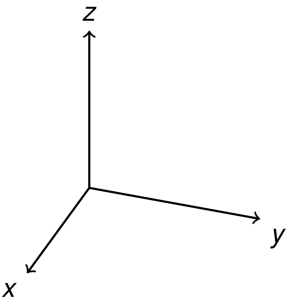




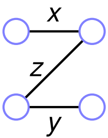
Maximum matching and simplex.



$$\max x + y + z$$
$$x \leq 1$$
$$x + z \leq 1$$
$$z + y \leq 1$$
$$y \leq 1$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$



Maximum matching and simplex.



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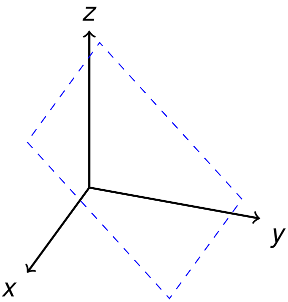
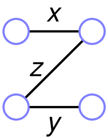


Blue constraints intersect.

Maximum matching and simplex.



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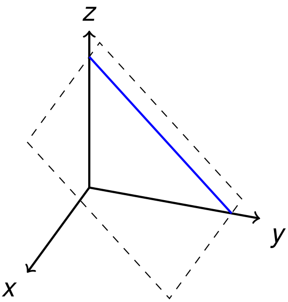
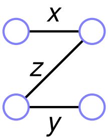
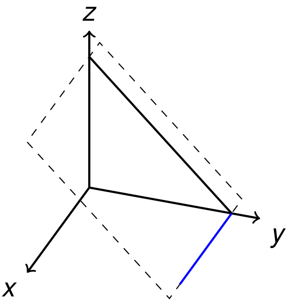


Blue constraints intersect.

Maximum matching and simplex.

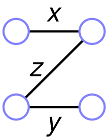


$$
\begin{aligned}
\max x + y + z \\
x \le 1 \\
x + z \le 1 \\
z + y \le 1 \\
y \le 1 \\
x \ge 0 \\
y \ge 0 \\
z \ge 0
\end{aligned}
$$



Blue constraints intersect.

Maximum matching and simplex.



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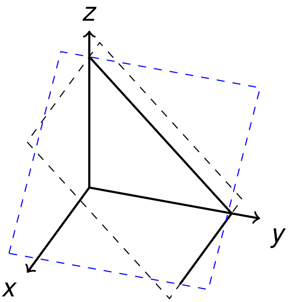
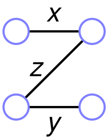


Blue constraints intersect.

Maximum matching and simplex.



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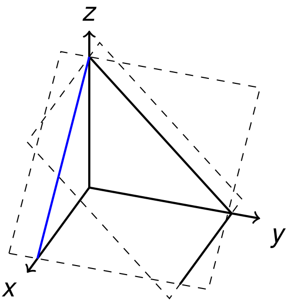
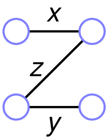


Blue constraints intersect.

Maximum matching and simplex.



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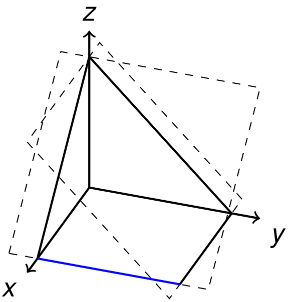
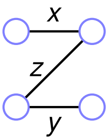


Blue constraints intersect.

Maximum matching and simplex.



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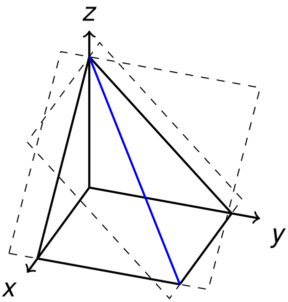
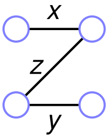
\end{aligned}
$$



Blue constraints intersect.

Maximum matching and simplex.



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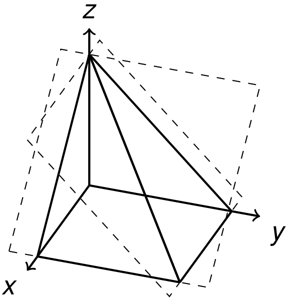
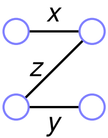


Blue constraints intersect.

Maximum matching and simplex.



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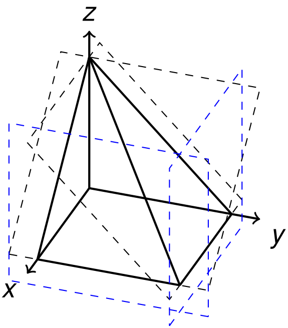

Blue constraints redundant.

Maximum matching and simplex.

$$\max x + y + z$$

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$$x \geq 0$$
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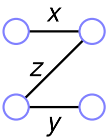
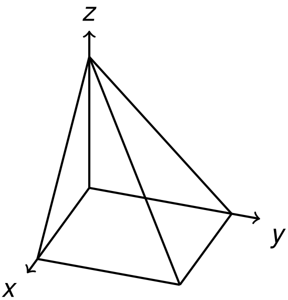
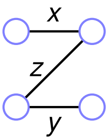
$$z \geq 0$$





Maximum matching and simplex.



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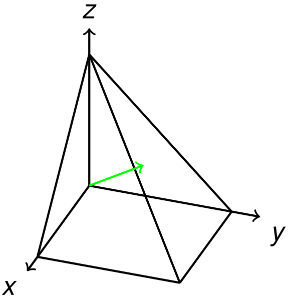
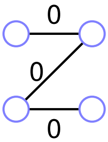


Maximum matching and simplex.



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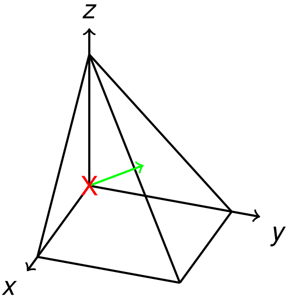
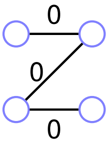


Blue constraints tight.

Maximum matching and simplex.



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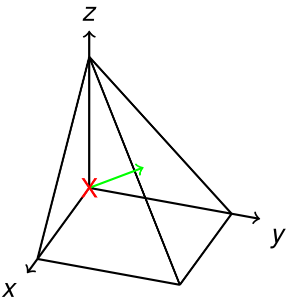

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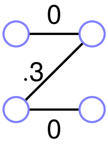

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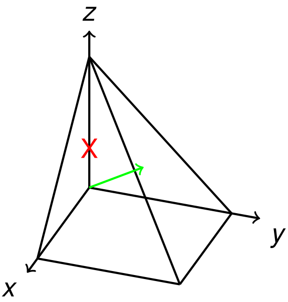

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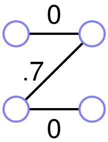

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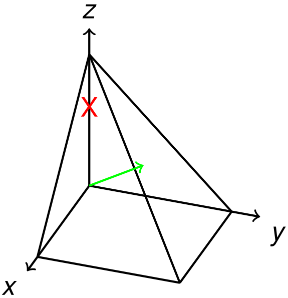

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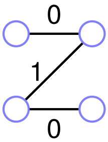

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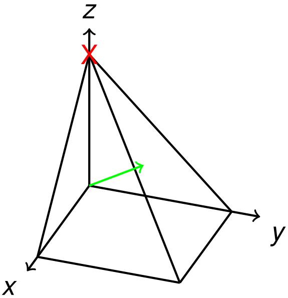

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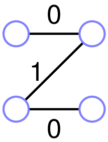

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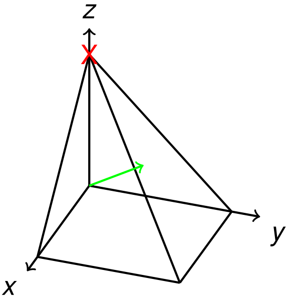

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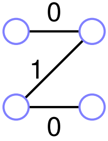

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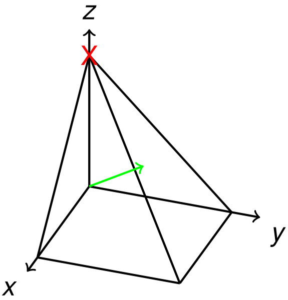

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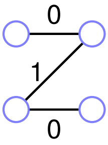

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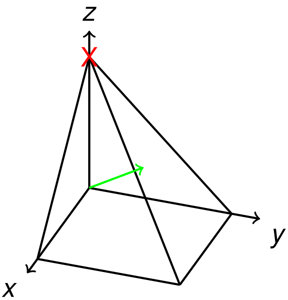

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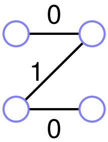

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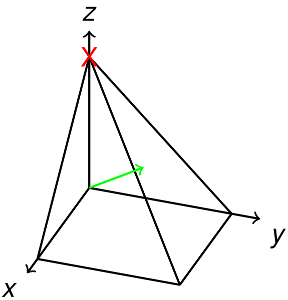

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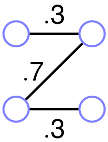

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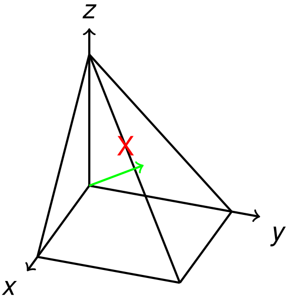

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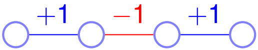
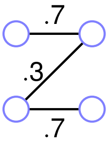


Augmenting Path.

Maximum matching and simplex.



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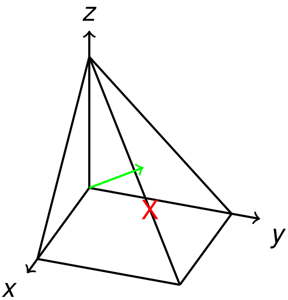

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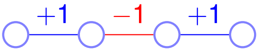
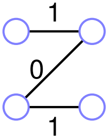


Augmenting Path. Via Gaussian Elimination!

Maximum matching and simplex.

$$\max x + y + z$$



$$x + z \leq 1$$
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$$x \geq 0$$
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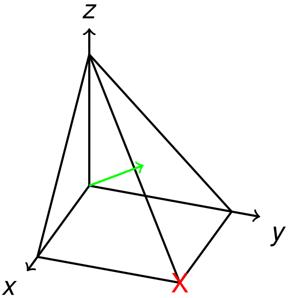

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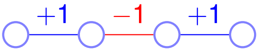

Augmenting Path. Via Gaussian Elimination!

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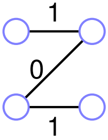

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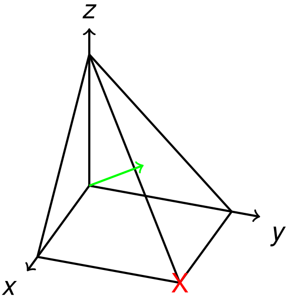

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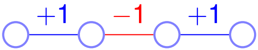

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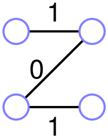
Maximum matching and simplex.



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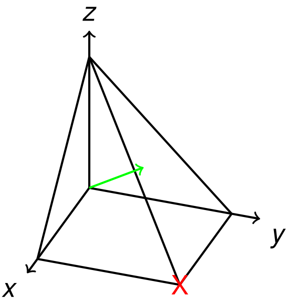
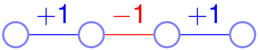
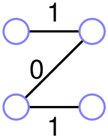


Blue constraints tight.



Augmenting Path. Via Gaussian Elimination!

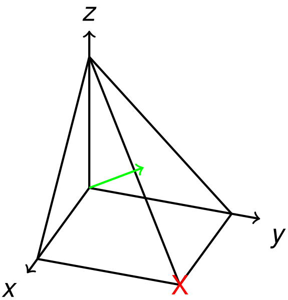
Maximum matching and simplex.



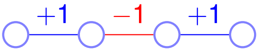
$$\max x + y + z$$

$$x + z \le 1 \quad a$$
$$z + y \le 1 \quad b$$

$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0 \quad c$$



Blue constraints tight.



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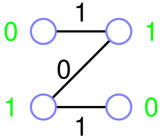


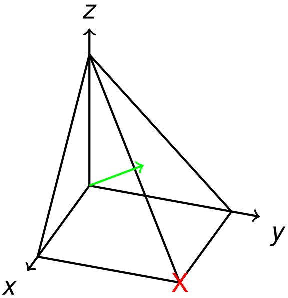
$x + z \leq 1 \quad a = 1$

$z + y \leq 1 \quad b = 1$

$x \geq 0$

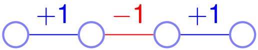
$y \geq 0$

$z \geq 0 \quad c = 1$



Blue constraints tight.

Sum: $x + 2z + y$.



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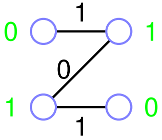

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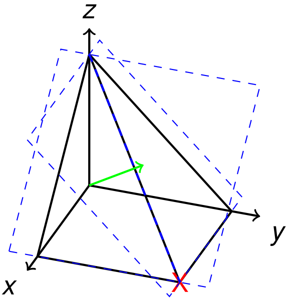

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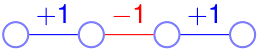

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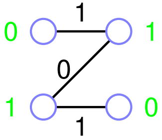

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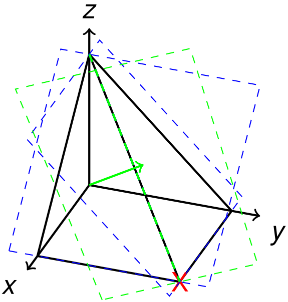

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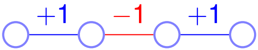

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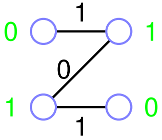

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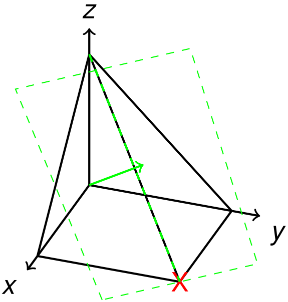

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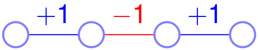

Augmenting Path. Via Gaussian Elimination!

Convex Separator.

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Farkas

Convex Separator.

Farkas

Strong Duality!!!!!

Convex Separator.

Farkas

Strong Duality!!!!! Maybe.

Linear Equations.

$Ax = b$

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A is $n \times n$ matrix...

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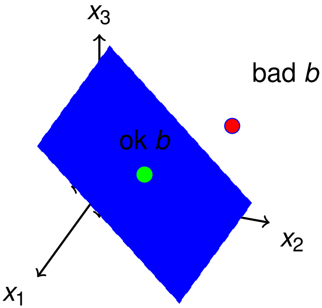

Strong Duality.

Strong Duality.

Later.

Strong Duality.

Later. Actually. No.

Strong Duality.

Later. Actually. No. Now

Strong Duality.

Later. Actually. No. Now ...ish.

Special Cases:

Strong Duality.

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Special Cases:

min-max 2 person games and experts.

Strong Duality.

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- Special Cases:
  - min-max 2 person games and experts.
  - Max weight matching and algorithm.

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  - Approximate: facility location primal dual.

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min-max 2 person games and experts.

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Approximate: facility location primal dual.

Today: Geometry!

Convex Body and point.

For a convex body P and a point b, $b \in P$ or hyperplane separates P from b.

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v, α, where $v \cdot x \leq \alpha$ and $v \cdot b > \alpha$.

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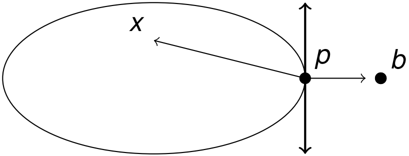
point p where $(x-p)^T(b-p) < 0$



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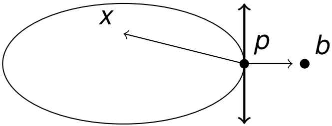

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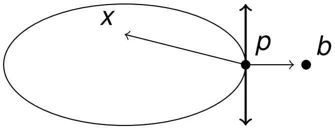

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Done

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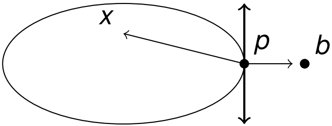

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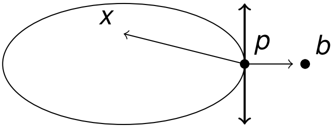

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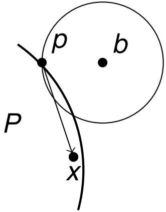

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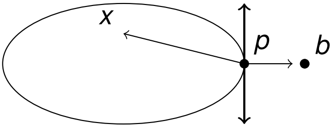

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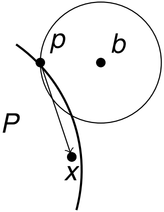

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$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x-p}$ and $\overrightarrow{b-p}$.

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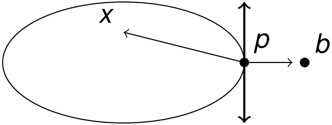

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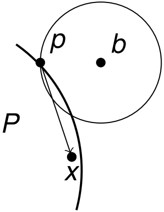

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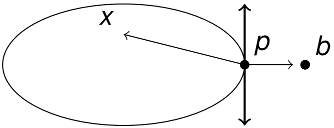

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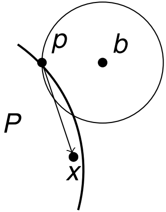

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Must be closer point b on line from p to x.

All points on line to x are in polytope.

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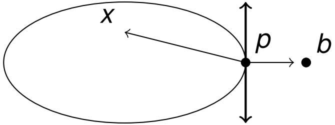

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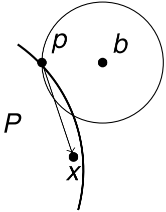

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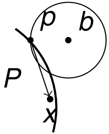
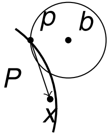
Contradicts choice of p as closest point to b in polytope.

More formally.



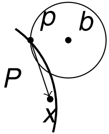
Squared distance to b from $p + (x - p)\mu$

More formally.



Squared distance to b from $p+(x-p)\mu$
point between p and x

More formally.



Squared distance to b from $p+(x-p)\mu$

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$$(|p-b|-\mu|x-p|cos\theta)^2+(\mu|x-p|sin\theta)^2$$

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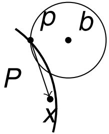

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$(|p-b|-\mu|x-p|cos\theta)^2+(\mu|x-p|sin\theta)^2$

θ is the angle between $x-p$ and $b-p$.

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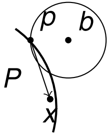

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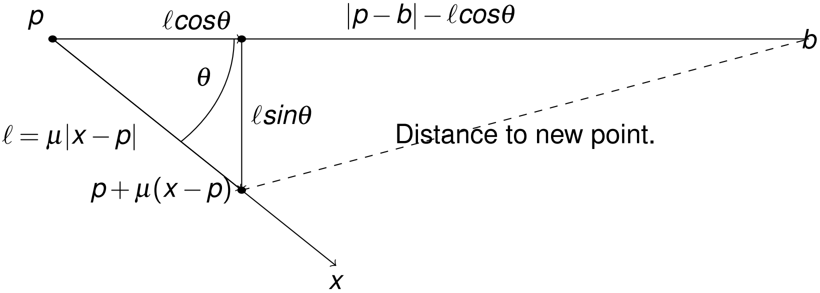

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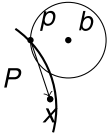

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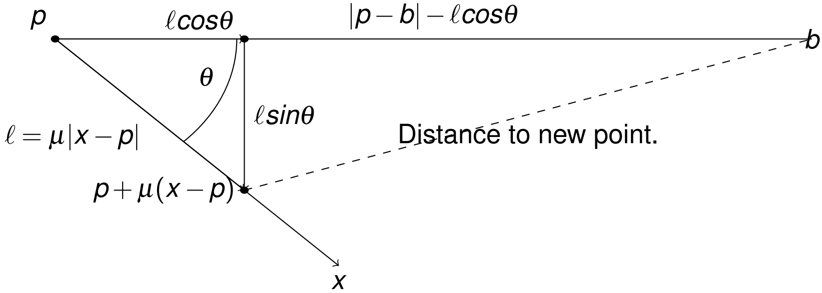
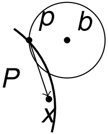


Simplify:

More formally.



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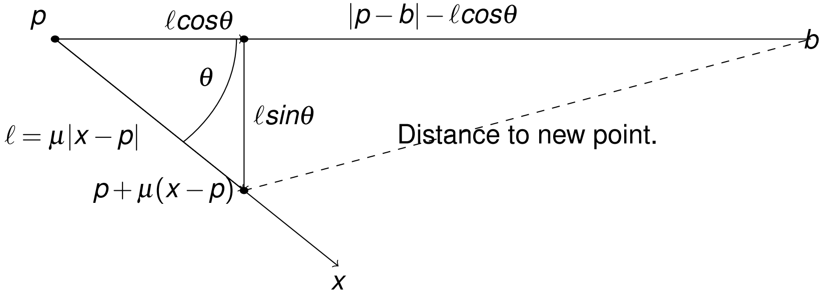
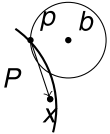


Simplify:
$|p-b|^2-2\mu|p-b||x-p|cos\theta+(\mu|x-p|)^2$.

More formally.



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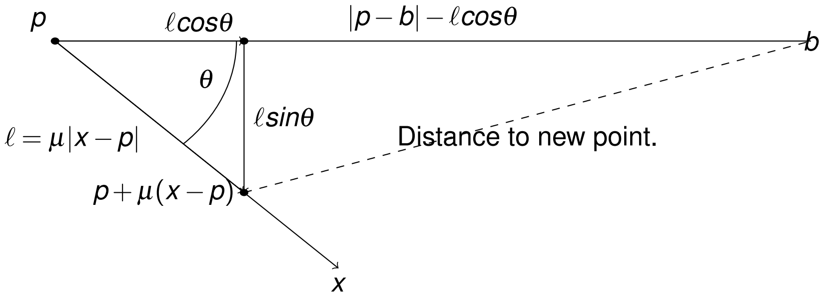
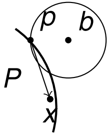


Simplify:
$|p-b|^2-2\mu|p-b||x-p|cos\theta+(\mu|x-p|)^2$.
Derivative with respect to μ ...

More formally.



Squared distance to b from $p + (x - p)\mu$

point between p and x

$(|p - b| - \mu|x - p|cos\theta)^2 + (\mu|x - p|sin\theta)^2$

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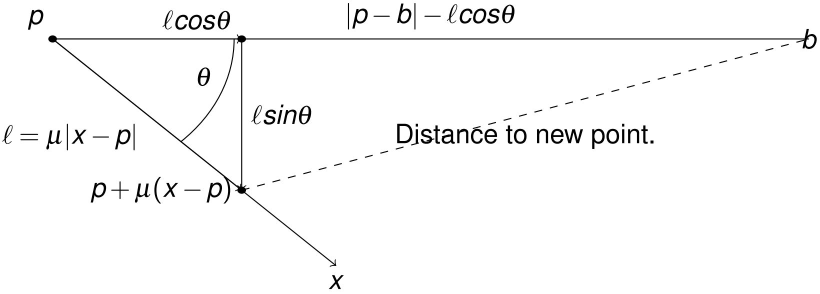


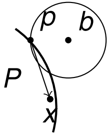
Simplify:

$|p - b|^2 - 2\mu|p - b||x - p|cos\theta + (\mu|x - p|)^2$.

Derivative with respect to μ ...

$-2|p - b||x - p|cos\theta + 2(\mu|x - p|^2)$.

More formally.



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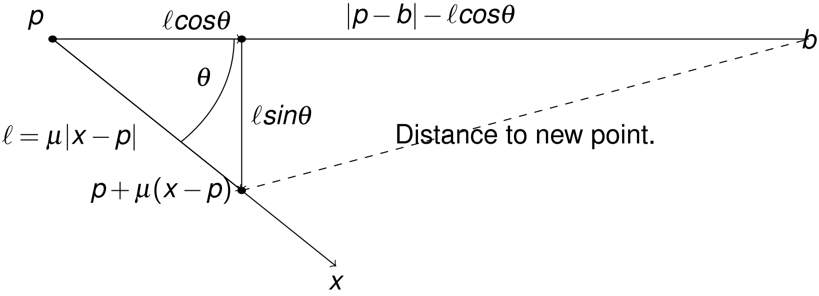


Simplify:

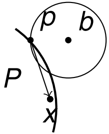
$|p-b|^2-2\mu|p-b||x-p|cos\theta+(\mu|x-p|)^2.$

Derivative with respect to μ ...

$-2|p-b||x-p|cos\theta+2(\mu|x-p|^2).$

which is negative for a small enough value of μ

More formally.

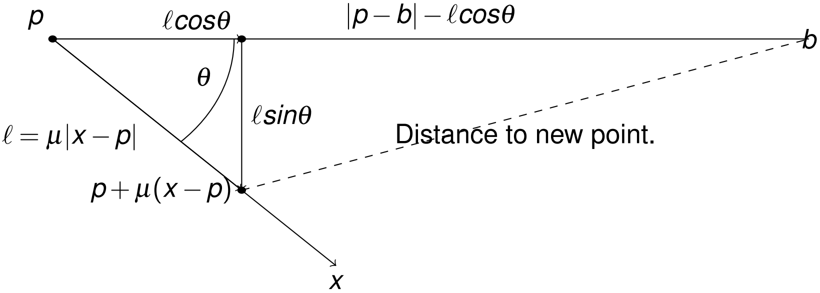


Squared distance to b from $p+(x-p)\mu$

point between p and x

$(|p-b|-\mu|x-p|cos\theta)^2+(\mu|x-p|sin\theta)^2$

θ is the angle between $x-p$ and $b-p$.



Simplify:

$|p-b|^2-2\mu|p-b||x-p|cos\theta+(\mu|x-p|)^2$.

Derivative with respect to μ ...

$-2|p-b||x-p|cos\theta+2(\mu|x-p|^2)$.

which is negative for a small enough value of μ (for positive $cos\theta$.)

Generalization: exercise.

Theorems of Alternatives.

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y is normal.

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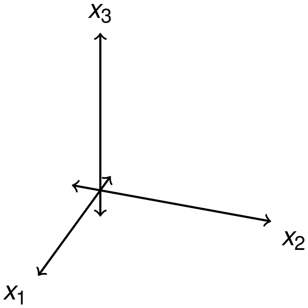
There is a separating hyperplane between any two convex bodies.

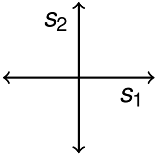
Idea: Let closest pair of points in two bodies define direction.

$Ax = b,\ x \geq 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Coordinates $s = b - Ax$.
$x \geq 0$ where $s = 0$?



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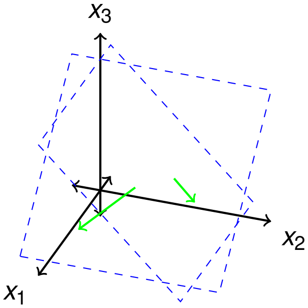

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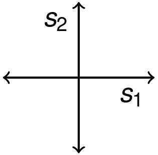

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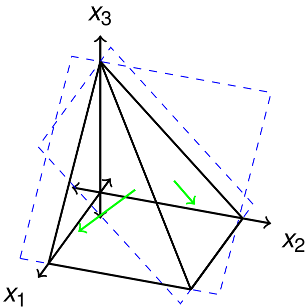

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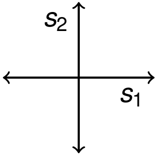

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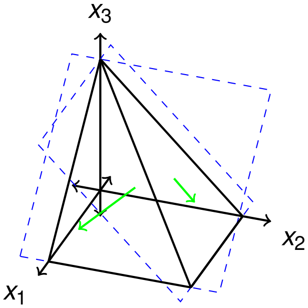

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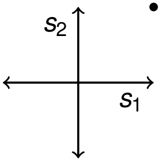

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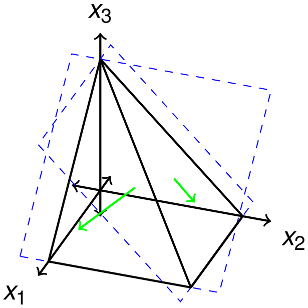

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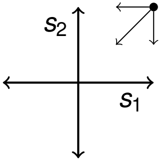

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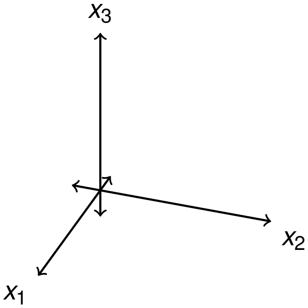

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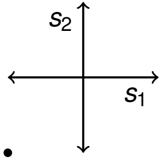

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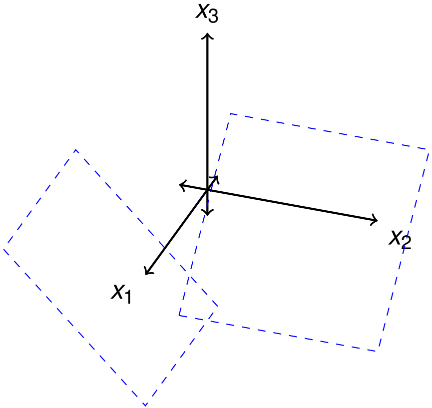

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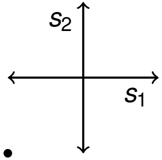

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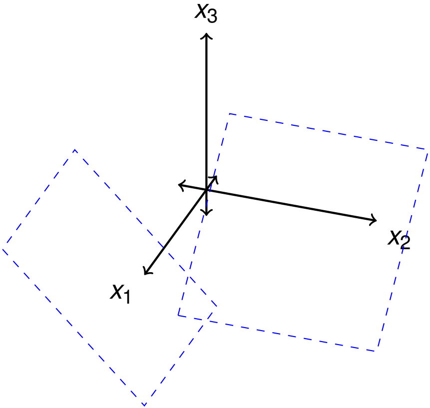

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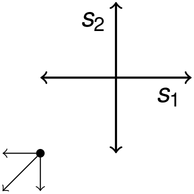

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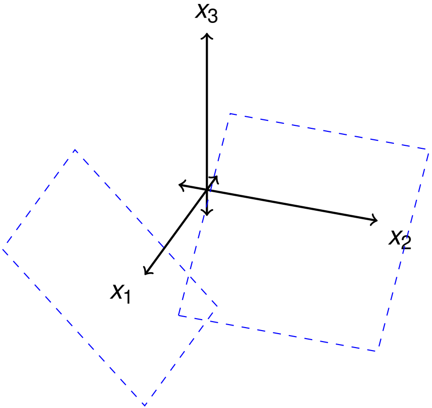

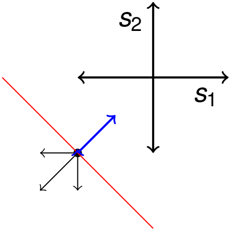

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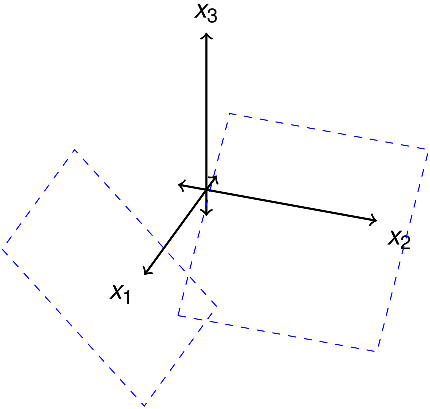

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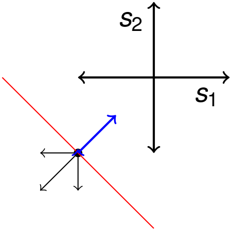

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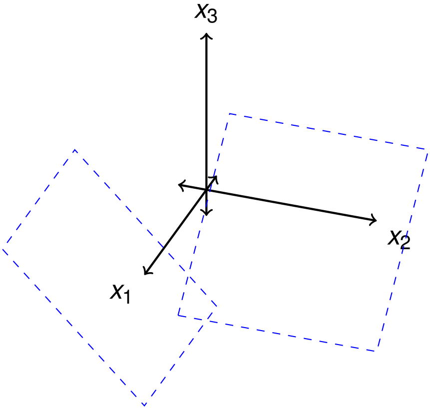

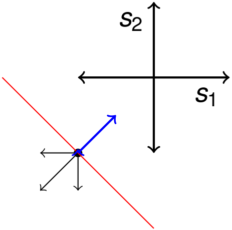

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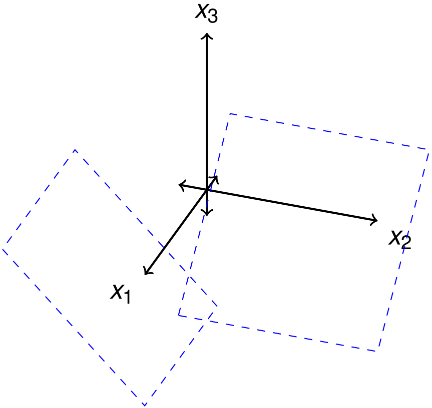

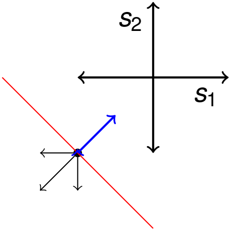

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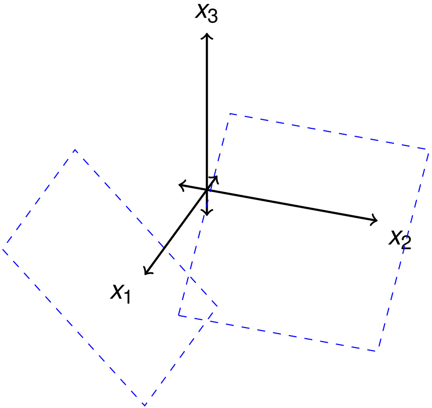

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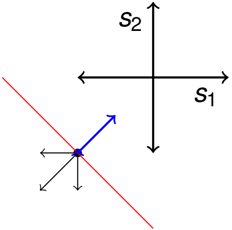

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Farkas 2

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(1) $Ax \leq b$

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Strong Duality

(From Goemans notes.)

$$\text{Primal P} \quad z^* = \min c^T x$$
$$Ax = b$$
$$x \geq 0$$

$$\text{Dual D} : w^* = \max b^T y$$
$$A^T y \leq c$$

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Weak Duality: x, y- feasible P, D: $x^T c \geq b^T y$.

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$$\begin{aligned} x^T c - b^T y &= x^T c - x^T A^T y \\ &= x^T (c - A^T y) \\ &\geq 0 \end{aligned}$$

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

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Primal feasible, bounded, value z^*.

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$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0$$

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Case 1: $\lambda > 0$.

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$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$.

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Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

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Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^T x < 0$.

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$$\begin{pmatrix} c^T & -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^T x < 0$.
Feasible $\tilde{x}$ for Primal.

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value z^*.

Claim: Exists a solution to dual of value at least z^*.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want y.

$$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.$$

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Primal unbounded!

See you on Tuesday.