Crazy Picture.

Maximum matching and simplex.

$\max x + y + z$

$x \leq 1$

$x + z \leq 1 \quad a = 1$

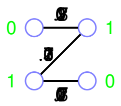
$z + y \leq 1 \quad b = 1$

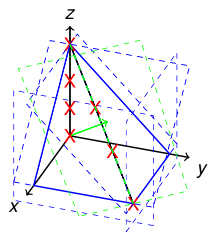
$y \leq 1$

$x \geq 0$

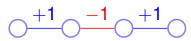
$y \geq 0$

$z \geq 0 \quad c = 1$





Sum: $x + 2z + y$.



Augmenting Path. Via Gaussian Elimination!

Convex Separator.

Farkas

Strong Duality!!!!! Maybe.

Linear Equations.

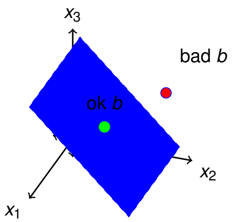
$Ax = b$

A is $n \times n$ matrix...

..has a solution.

If rows of A are linearly independent.
 $y^T A \neq 0$ for any y

..or if b in subspace of A.



Strong Duality.

Later. Actually. No. Now ...ish.
 Special Cases:
 min-max 2 person games and experts.
 Max weight matching and algorithm.
 Approximate: facility location primal dual.

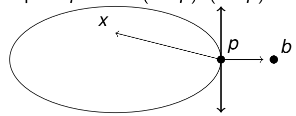
Today: Geometry!

Convex Body and point.

For a convex body P and a point b, $b \in P$ or hyperplane separates P from b.

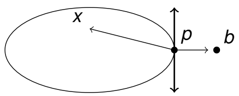
v, α, where $v \cdot x \leq \alpha$ and $v \cdot b > \alpha$.

point p where $(x - p)^T (b - p) < 0$



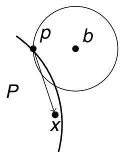
Proof.

For a convex body P and a point b, $b \in A$ or there is point p where $(x-p)^T(b-p) < 0$



Proof: Choose p to be closest point to b in P.

Done or $\exists\, x \in P$ with $(x-p)^T(b-p) \geq 0$



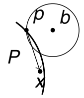
$(x-p)^T(b-p) \geq 0$
$\rightarrow \leq 90^\circ$ angle between $\overrightarrow{x-p}$ and $\overrightarrow{b-p}$.

Must be closer point b on line from p to x.

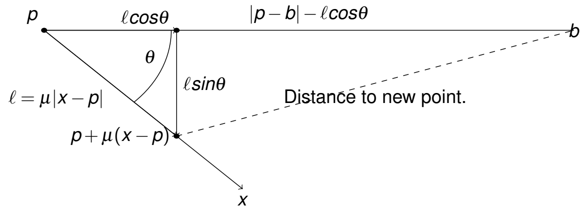
All points on line to x are in polytope.

Contradicts choice of p as closest point to b in polytope.

More formally.



Squared distance to b from $p+(x-p)\mu$
point between p and x
$(|p-b| - \mu|x-p|cos\theta)^2 + (\mu|x-p|sin\theta)^2$
θ is the angle between $x-p$ and $b-p$.



Simplify:
$|p-b|^2 - 2\mu|p-b||x-p|cos\theta + (\mu|x-p|)^2$.
Derivative with respect to μ ...
$-2|p-b||x-p|cos\theta + 2(\mu|x-p|^2)$.
which is negative for a small enough value of μ (for positive $cos\theta$.)

Generalization: exercise.

Theorems of Alternatives.

Linear Equations: There is a separating hyperplane between a point and an affine subspace not containing it.

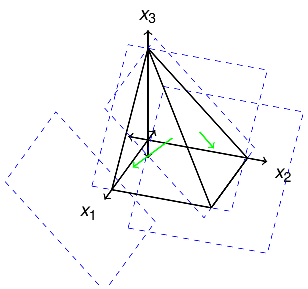
From $Ax = b$ use row reduction to get, e.g., $0 \neq 5$.
That is, find y where $y^TA = 0$ and $y^Tb \neq 0$.
Space is image of A. Affine subspace is columnspan of A.
y is normal. y in nullspace for column span.
$y^Tb \neq 0 \implies b$ not in column span.

There is a separating hyperplane between any two convex bodies.

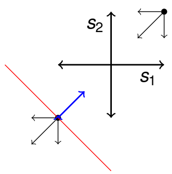
Idea: Let closest pair of points in two bodies define direction.

$Ax = b$, $x \geq 0$

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} x = \begin{bmatrix}1\\ 1\end{bmatrix}$$



Coordinates $s = b - Ax$.
$x \geq 0$ where $s = 0$?



y where $y^T(b-Ax) < y^T(0) < 0$ for all $x \geq 0$ $\rightarrow y^Tb < 0$ and $y^TA \geq 0$.

Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
(2) $y^TA \geq 0, y^Tb < 0$.

Farkas 2

Farkas A: Solution for exactly one of:
(1) $Ax = b, x \geq 0$
(2) $y^TA \geq 0, y^Tb < 0$.

Farkas B: Solution for exactly one of:
(1) $Ax \leq b$
(2) $y^TA = 0, y^Tb < 0, y \geq 0$.

Strong Duality

(From Goemans notes.)

Primal P $\quad z^* = \min c^Tx$
$Ax = b$
$x \geq 0$

Dual D : $w^* = \max b^Ty$
$A^Ty \leq c$

Weak Duality: x, y- feasible P, D: $x^Tc \geq b^Ty$.

$$\begin{aligned} x^Tc - b^Ty &= x^Tc - x^TA^Ty \\ &= x^T(c - A^Ty) \\ &\geq 0 \end{aligned}$$

Strong duality If P or D is feasible and bounded then $z^* = w^*$.

Primal feasible, bounded, value z^*.

Claim: Exists a solution to dual of value at least z^*.

$\exists y, y^T A \leq c, b^T y \geq z^*$.

Want y.

$$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ -z^* \end{pmatrix}.$$

If none, then Farkas B says

$\exists x, \lambda \geq 0$.

$$\begin{pmatrix} A & -b \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = 0 \qquad \begin{pmatrix} c^T & -z^* \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} < 0$$

$\exists x, \lambda$ with $Ax - b\lambda = 0$ and $c^t x - z^* \lambda < 0$

Case 1: $\lambda > 0$. $A(\frac{x}{\lambda}) = b$, $c^T(\frac{x}{\lambda}) < z^*$. Better Primal!!

Case 2: $\lambda = 0$. $Ax = 0, c^T x < 0$.
Feasible $\tilde{x}$ for Primal.
(a) $\tilde{x} + \mu x \geq 0$ since $\tilde{x}, x, \mu \geq 0$.
(b) $A(\tilde{x} + \mu x) = A\tilde{x} + \mu A x = b + \mu \cdot 0 = b$. Feasible
$c^T(\tilde{x} + \mu x) = x^T \tilde{x} + \mu c^T x \to -\infty$ as $\mu \to \infty$
Primal unbounded!

See you on Tuesday.