Today

Routing and Experts.

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Review: linear programming.

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Taking Duals.

Toll/Congestion

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Given $(s_1, t_1) \dots (s_k, t_k)$.
Row: choose routing of all paths.
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Runtime only dependent on m and T (number of days.)

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$G^* = c_{\max}$ — Best row payoff against average routing.

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$\rightarrow c_{max} - C^* \leq 2\varepsilon k$ □

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Homework: $O(km \log n)$ algorithm.

Fractional versus Integer.

Did we solve path routing?

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Yes?

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Yes? No?

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No!

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Homework 2. Problem 1.

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Homework 2. Problem 1.

Decent solution to path routing problem?

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For each s_i, t_i, choose path p_i with probability $f(p_i)$.

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Concentration results?

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$\rightarrow E(\tilde{c}(e)) = c(e)$.

Concentration (law of large numbers)
 $c(e)$ is relatively large ($\Omega(\log n)$)
 $\rightarrow \tilde{c}(e) \approx c(e)$.

Concentration results? later.

Profit maximization.

Plant Carrots or Peas?

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.

Garden has 60 yards of sunny land and 75 yards of shady land.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.

Garden has 60 yards of sunny land and 75 yards of shady land.

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.

Garden has 60 yards of sunny land and 75 yards of shady land.

To pea or not to pea, that is the question!

To pea or not to pea.

4$ for peas.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1 - to pea!

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

To pea or not to pea.

4\$ for peas. 2\$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.
x_1- to pea! x_2 to carrot ?
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.
$2x_1 \leq 40$
Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can't make negative!

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$$3x_1 + 2x_2 \leq 100$$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can't make negative! $x_1, x_2 \geq 0$.

To pea or not to pea.

4$ for peas. 2$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

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Peas 2 yards/bushel of sunny land. Have 40 sq yards.

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Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can't make negative! $x_1, x_2 \geq 0$.

A linear program.

To pea or not to pea.

4\$ for peas. 2\$ bushel of carrots.

x_1- to pea! x_2 to carrot ?

Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.

Carrots take 2 unit of water/bushel.

Peas take 3 units of water/bushel. 100 units of water.

$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.

$2x_1 \leq 40$

Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.

$3x_2 \leq 75$

Can't make negative! $x_1, x_2 \geq 0$.

A linear program.

$$
\begin{aligned}
\max 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\end{aligned}
$$

Optimal point?

$$
\begin{aligned}
\max 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\end{aligned}
$$

Optimal point?

Try every point

$$
\begin{aligned}
\max 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\end{aligned}
$$

Optimal point?

Try every point if we only had time!

$$
\begin{aligned}
\max 4x_1 + 2x_2 \\
2x_1 \leq 40 \\
3x_2 \leq 75 \\
3x_1 + 2x_2 \leq 100 \\
x_1, x_2 \geq 0
\end{aligned}
$$

Optimal point?

Try every point if we only had time!

How many points?

$$\max 4x_1 + 2x_2$$
$$2x_1 \leq 40$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

Optimal point?

Try every point if we only had time!

How many points?

Real numbers?

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

Optimal point?

Try every point if we only had time!

How many points?

Real numbers?

Infinite.

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

Optimal point?

Try every point if we only had time!

How many points?

Real numbers?

Infinite. Uncountably infinite!

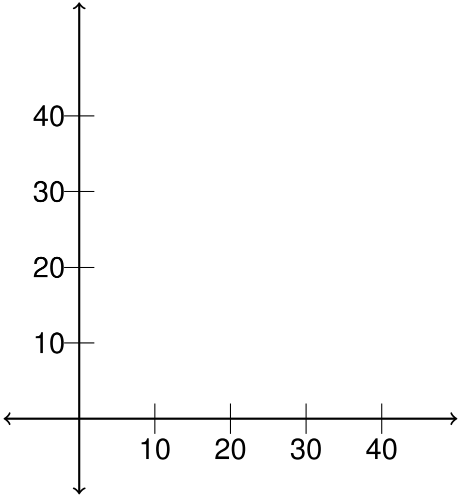
A linear program.

A linear program.

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

A linear program.

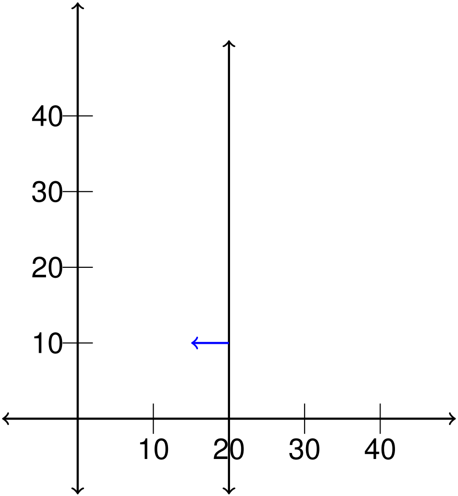
$$\max 4x_1 + 2x_2$$
$$2x_1 \leq 40$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$



Optimal point?

A linear program.

$$\max 4x_1 + 2x_2$$
$$2x_1 \leq 40$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$



Optimal point?

A linear program.

$$\max 4x_1 + 2x_2$$
$$2x_1 \leq 40$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
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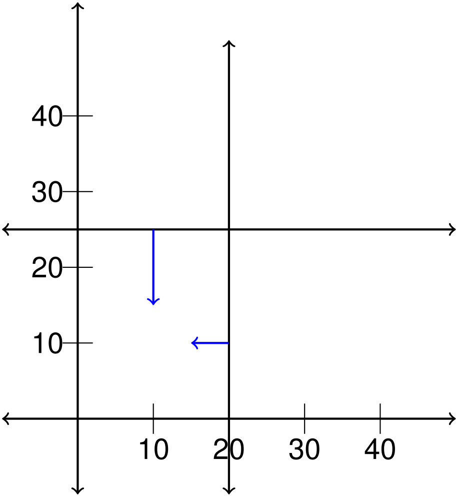

Optimal point?

A linear program.

$$\max 4x_1 + 2x_2$$
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$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
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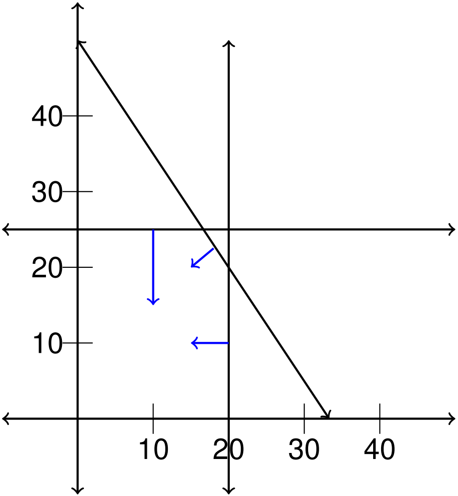
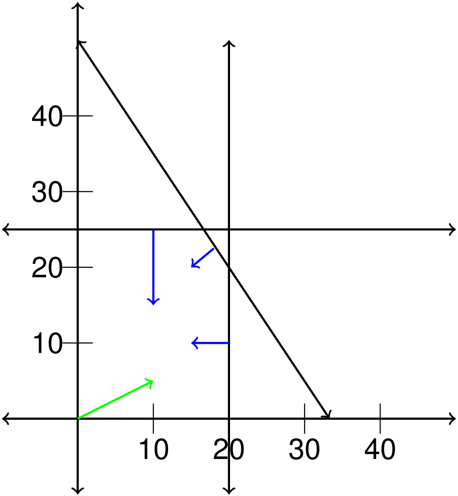


Optimal point?

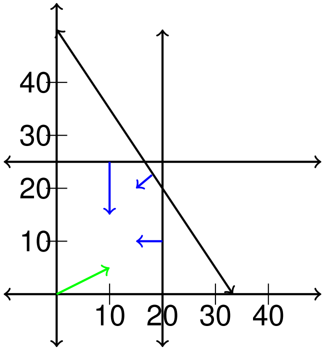
A linear program.

$$\max 4x_1 + 2x_2$$
$$2x_1 \leq 40$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

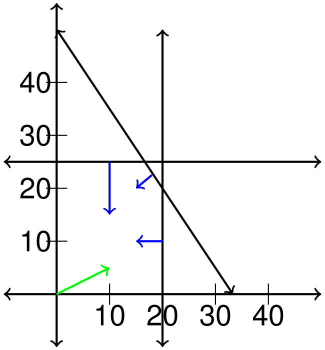


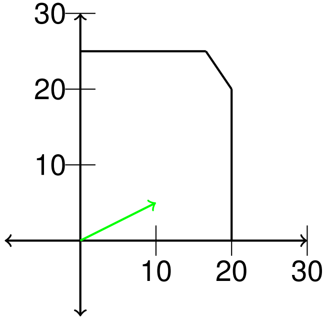
Optimal point?

Feasible Region.

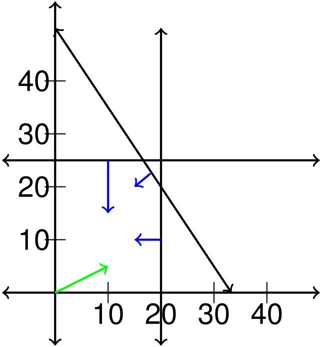


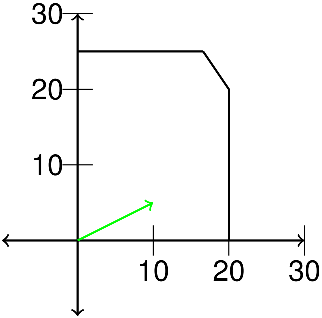
Feasible Region.





Feasible Region.

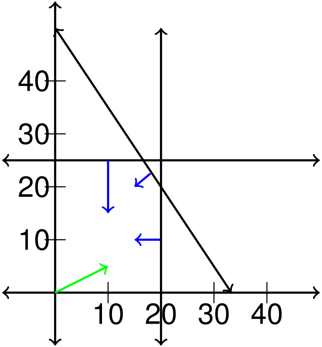


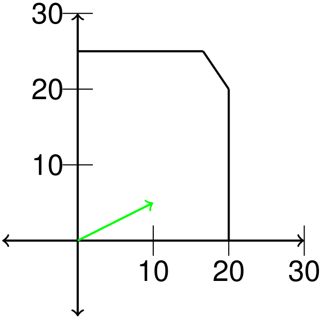


Convex.

Any two points in region connected by a line in region.

Feasible Region.



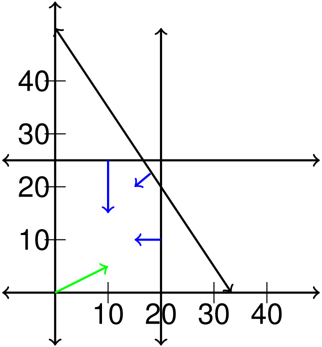


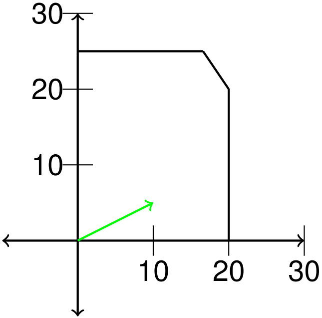
Convex.

Any two points in region connected by a line in region.

Algebraically:

Feasible Region.





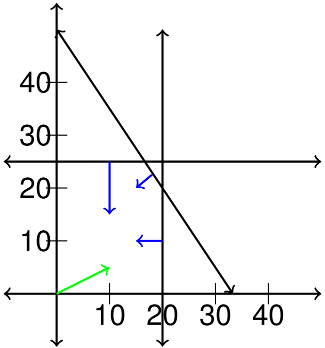
Convex.

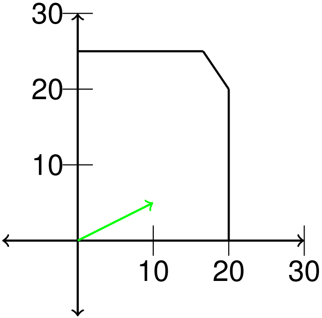
Any two points in region connected by a line in region.

Algebraically:

If x and x' satisfy onstraint,

Feasible Region.





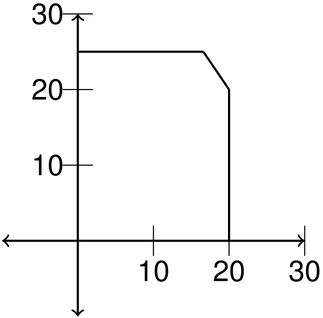
Convex.

Any two points in region connected by a line in region.

Algebraically:

If x and x' satisfy onstraint,

$\rightarrow x'' = \alpha x + (1-\alpha)x$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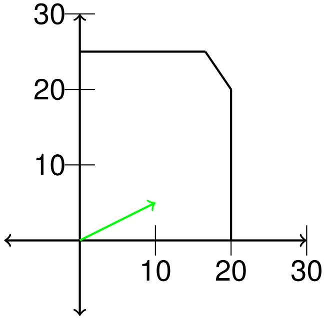
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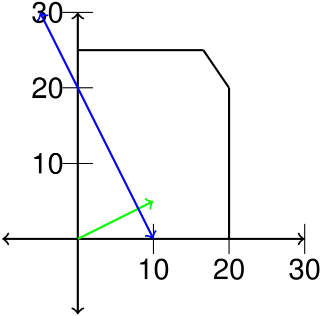

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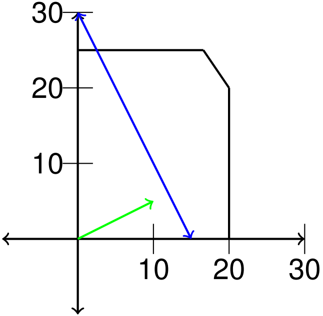
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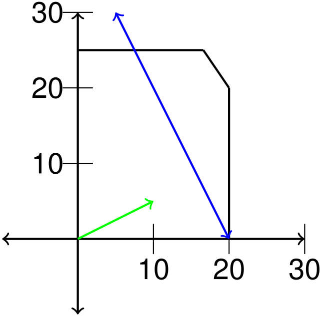
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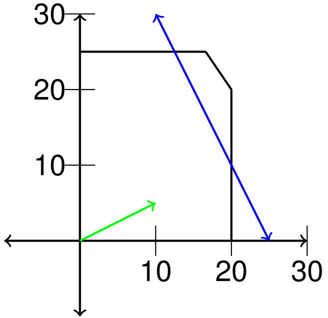

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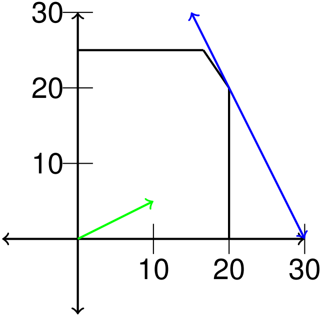
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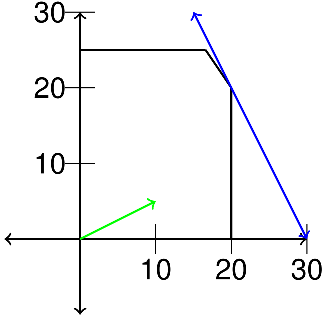
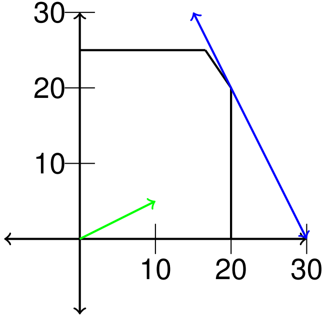
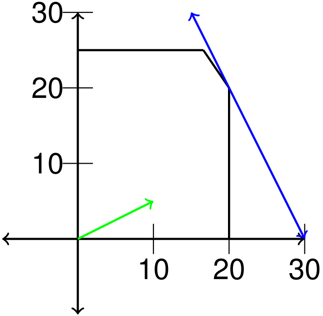


Optimal at pointy part of feasible region!



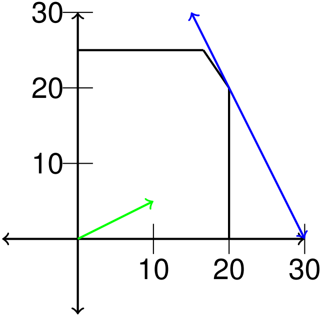
Optimal at pointy part of feasible region!
Vertex of region.



Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

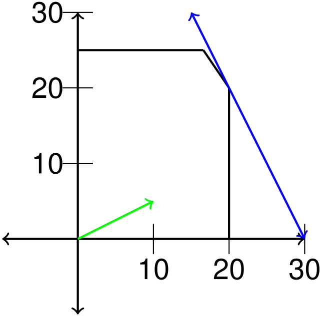


Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex!

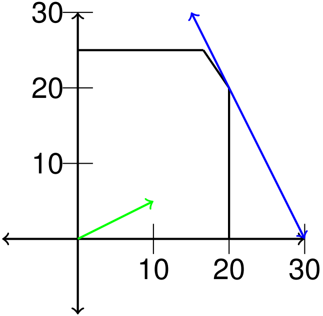


Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex! Choose best.



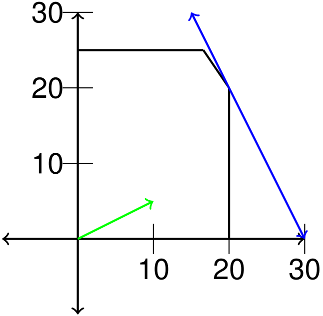
Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex! Choose best.

$O(m^2)$ if m constraints and 2 variables.



Optimal at pointy part of feasible region!

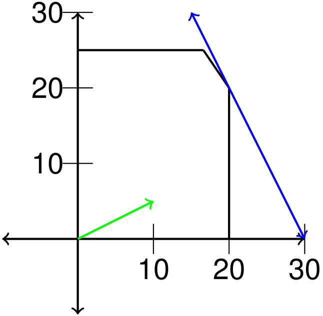
Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex! Choose best.

$O(m^2)$ if m constraints and 2 variables.

For n variables, m constraints, how many?



Optimal at pointy part of feasible region!

Vertex of region.

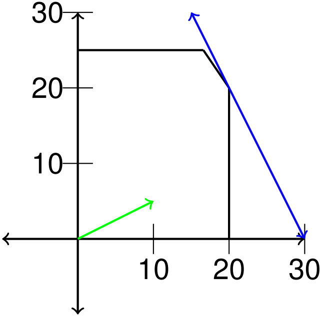
Intersection of two of the constraints (lines in two dimensions)!

Try every vertex! Choose best.

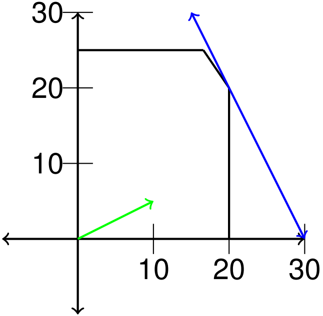
$O(m^2)$ if m constraints and 2 variables.

For n variables, m constraints, how many?

nm? $\binom{m}{n}$? $n+m$?



Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if m constraints and 2 variables.
For n variables, m constraints, how many?
nm? $\binom{m}{n}$? $n+m$?
$\binom{m}{n}$



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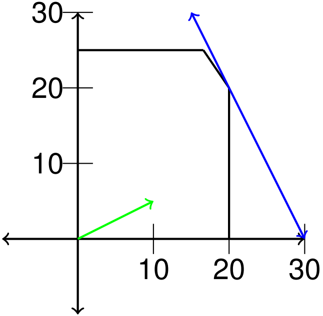

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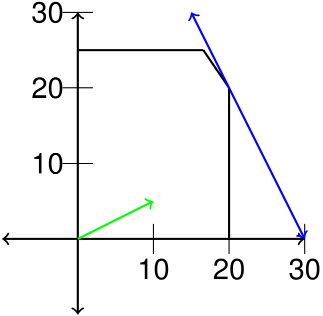
Finite!!!!!!



Optimal at pointy part of feasible region!

Vertex of region.

Intersection of two of the constraints (lines in two dimensions)!

Try every vertex! Choose best.

$O(m^2)$ if m constraints and 2 variables.

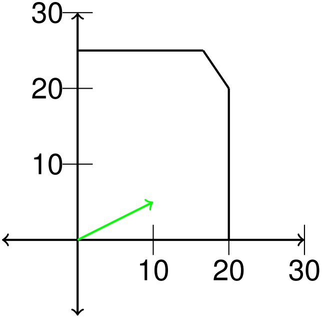
For n variables, m constraints, how many?

nm? $\binom{m}{n}$? $n+m$?

$\binom{m}{n}$

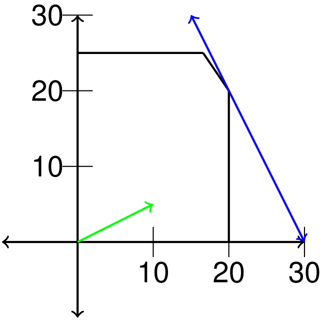
Finite!!!!!!

Exponential in the number of variables.



Simplex: Start at vertex.

$$\max 4x_1 + 2x_2$$
$$3x_1 \leq 60$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$



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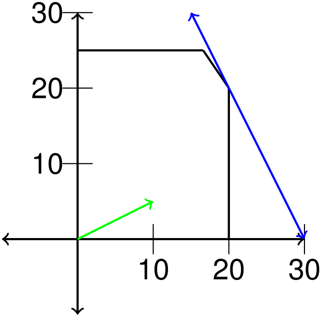
Simplex: Start at vertex. Move to better neighboring vertex.



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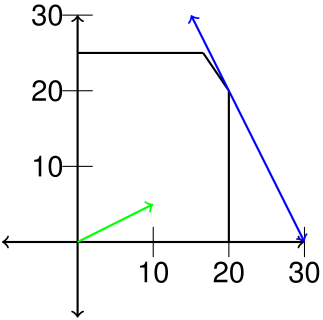
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor.



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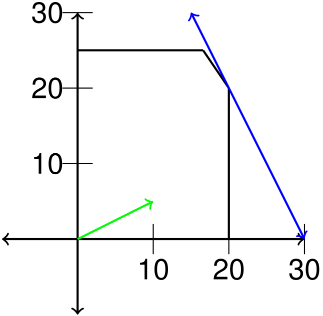
Until no better neighbor. This example.



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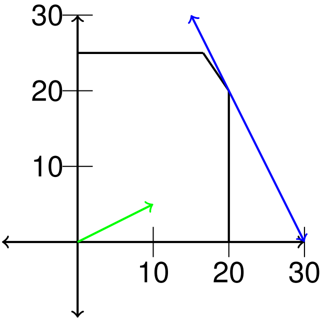
Until no better neighbor. This example.
(0,0) objective 0.



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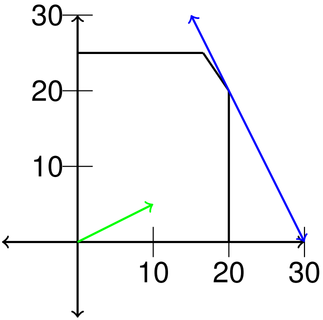
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. This example.
(0,0) objective 0. $\rightarrow$ $(0,25)$ objective 50.



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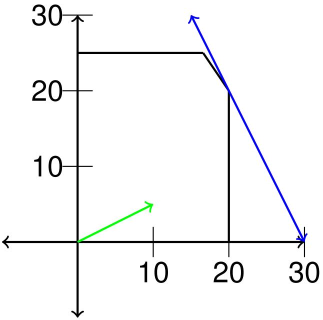
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. This example.
(0,0) objective 0. $\rightarrow (0,25)$ objective 50.
$\rightarrow (16\frac{2}{3}, 25)$ objective $115\frac{2}{3}$



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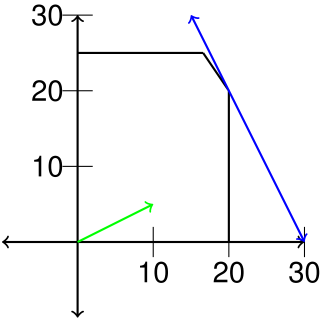
$$x_1, x_2 \ge 0$$

Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. This example.
(0,0) objective 0. $\rightarrow (0,25)$ objective 50.
$\rightarrow (16\frac{2}{3}, 25)$ objective $115\frac{2}{3}$
$\rightarrow (20,20)$ objective 120.



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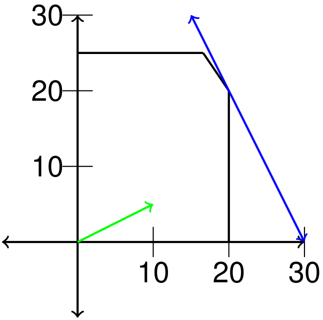
$\rightarrow (20, 20)$ objective 120.
Duality:



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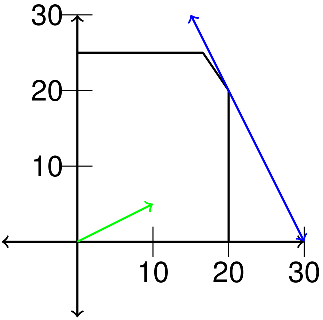
Duality:
Add blue equations to get objective function?



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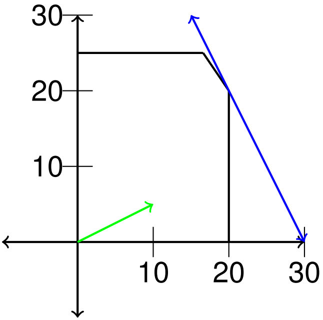
$\rightarrow (20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.



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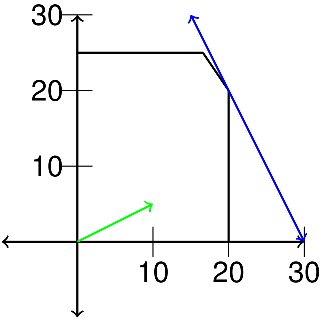
$\rightarrow (20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$.



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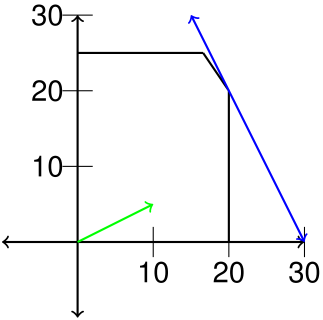
$\rightarrow (20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!



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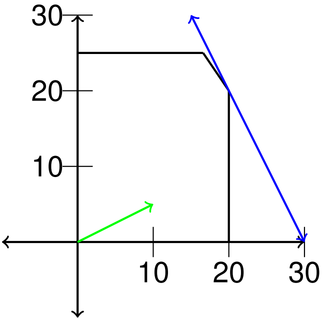
$\rightarrow (20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Objective value: 120.
Can we do better?



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$\rightarrow$ $(16\frac{2}{3}, 25)$ objective $115\frac{2}{3}$
$\rightarrow$ $(20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Objective value: 120.
Can we do better? No!
Dual problem: add equations to get best upper bound.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\le 4 \\ x_2 &\le 3 \\ x_1 + 2x_2 &\le 7 \\ x_1, x_2 &\ge 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$.

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Duality.

$$\max x_1 + 8x_2$$

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Best possible?

Duality.

$$\max x_1 + 8x_2$$

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One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

For any solution.
$x_1 \le 4$ and $x_2 \le 3$ ..

Duality.

$$\max x_1 + 8x_2$$

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..
....so $x_1 + 8x_2 \leq 4 + 8(3) = 28$.
Added equation 1 and 8 times equation 2
yields bound on objective..

Duality.

$$\max x_1 + 8x_2$$

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Better solution?
Better upper bound?

Duality.

$$\max x_1 + 8x_2$$

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Better solution?
Better upper bound?

Duality.

$$\begin{aligned} \max x_1 + 8x_2 & \\ x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.

Duality.

$$\max x_1 + 8x_2$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0$$

Solution value: 25.

Add equation 1 and 8 times equation 2 gives..

Duality.

$$\begin{aligned} \max x_1 + 8x_2 \\ x_1 \le 4 \\ x_2 \le 3 \\ x_1 + 2x_2 \le 7 \\ x_1, x_2 \ge 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \le 4 + 24 = 28$.

Duality.

$$\max x_1 + 8x_2$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28.$

Better way to add equations to get bound on function?

Duality.

$$\begin{aligned} \max x_1 + 8x_2 \\ x_1 \leq 4 \\ x_2 \leq 3 \\ x_1 + 2x_2 \leq 7 \\ x_1, x_2 \geq 0 \end{aligned}$$

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Add equation 1 and 8 times equation 2 gives..
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Better way to add equations to get bound on function?
Sure:

Duality.

$$\begin{aligned} \max x_1 + 8x_2 \\ x_1 \le 4 \\ x_2 \le 3 \\ x_1 + 2x_2 \le 7 \\ x_1, x_2 \ge 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \le 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

Duality.

$$\begin{aligned} \max x_1 + 8x_2 \\ x_1 \leq 4 \\ x_2 \leq 3 \\ x_1 + 2x_2 \leq 7 \\ x_1, x_2 \geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

$x_1 + 8x_2 \leq 6(3) + 7 = 25$.

Duality.

$$
\begin{aligned}
\max x_1 + 8x_2 \\
x_1 &\leq 4 \\
x_2 &\leq 3 \\
x_1 + 2x_2 &\leq 7 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

$$x_1 + 8x_2 \leq 6(3) + 7 = 25.$$

Thus, the value is at most 25.

Duality.

$$\begin{aligned} \max x_1 + 8x_2 \\ x_1 \leq 4 \\ x_2 \leq 3 \\ x_1 + 2x_2 \leq 7 \\ x_1, x_2 \geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

$$x_1 + 8x_2 \leq 6(3) + 7 = 25.$$

Thus, the value is at most 25.

The upper bound is same as solution!

Duality.

$$\begin{aligned} \max x_1 + 8x_2 & \\ x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.

$$x_1 + 8x_2 \leq 6(3) + 7 = 25.$$

Thus, the value is at most 25.

The upper bound is same as solution!

Proof of optimality!

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Will this always work?

Duality:example

Idea: Add up positive linear combination of inequalities to “get” upper bound on optimization function.

Will this always work?

How to find best upper bound?

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

The left hand side should "dominate" optimization function:

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
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The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

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The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...

$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$

The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..

$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

Find best y_i's to minimize upper bound?

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$

The dual, the dual, the dual.

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$$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

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A linear program.

The dual, the dual, the dual.

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A linear program.
The Dual linear program.

The dual, the dual, the dual.

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A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

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A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

The dual, the dual, the dual.

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The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1, 3)$; Dual: $(y_1, y_2, y_3) = (0, 6, 1)$.

Value of both is 25!

Primal is optimal ... and dual is optimal!

The dual.

In general.

Primal LP	Dual LP
max $c \cdot x$	min $y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

The dual.

In general.

Primal LP	Dual LP
max $c \cdot x$	min $y^T b$
$Ax \leq b$	$y^T A \geq c$
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Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

The dual.

In general.

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Weak Duality: primal (P) $\leq$ dual (D)

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Weak Duality: primal (P) $\leq$ dual (D)

Feasible (x, y)

The dual.

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Weak Duality: primal (P) $\leq$ dual (D)

Feasible (x, y)
$P(x)$

The dual.

In general.

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Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal (P) $\leq$ dual (D)

Feasible (x, y)
$P(x) = c \cdot x$

The dual.

In general.

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Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal (P) $\leq$ dual (D)

Feasible (x, y)

$P(x) = c \cdot x \leq y^T Ax$

The dual.

In general.

Primal LP	Dual LP
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$Ax \leq b$	$y^T A \geq c$
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Weak Duality: primal (P) $\leq$ dual (D)

Feasible (x, y)

$$P(x) = c \cdot x \leq y^T Ax \leq y^T b \cdot x = D(y).$$

The dual.

In general.

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
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$$P(x) = c \cdot x \leq y^T Ax \leq y^T b \cdot x = D(y).$$

Strong Duality: next lecture.

Complementary Slackness

Primal LP	Dual LP
max $c \cdot x$	min $y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Given A, b, c, and feasible solutions x and y.

Complementary Slackness

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Given A, b, c, and feasible solutions x and y.

Solutions x and y are both optimal if and only if
$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

Complementary Slackness

Primal LP	Dual LP
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$x_i(c_i - (y^T A)_i) = 0 \rightarrow$
$\sum_i (c_i - (y^T A)_i) x_i$

Complementary Slackness

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$\max c \cdot x$	$\min y^T b$
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$x_i(c_i - (y^T A)_i) = 0 \rightarrow$
$\sum_i (c_i - (y^T A)_i) x_i = cx - y^T Ax$

Complementary Slackness

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$\sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax$

Complementary Slackness

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$\sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \rightarrow by = y^T Ax$.

Complementary Slackness

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$cx = by$.

Complementary Slackness

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$cx = by$.

If both are feasible, $cx \leq by$, so must be optimal.

Complementary Slackness

Primal LP	Dual LP
max $c \cdot x$	min $y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Given A, b, c, and feasible solutions x and y.

Solutions x and y are both optimal if and only if
$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

$x_i(c_i - (y^T A)_i) = 0 \to$
$\sum_i (c_i - (y^T A)_i) x_i = cx - y^T Ax \to cx = y^T Ax$.

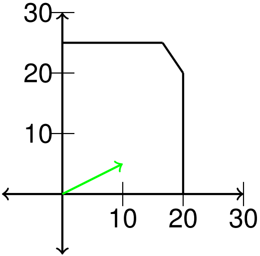
$y_j(b_j - (Ax)_j) = 0 \to$
$\sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \to by = y^T Ax$.

$cx = by$.

If both are feasible, $cx \leq by$, so must be optimal.

In words: nonzero dual variables only for tight constraints!

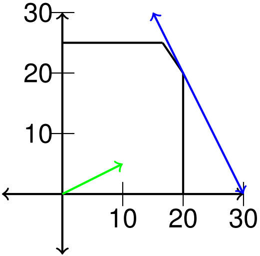
Again: simplex



Simplex: Start at vertex.

$$\max 4x_1 + 2x_2$$
$$3x_1 \leq 60$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

Again: simplex



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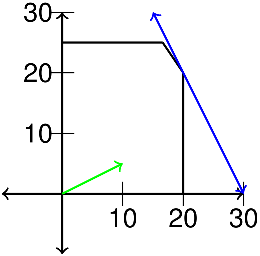
Simplex: Start at vertex. Move to better neighboring vertex.

Again: simplex



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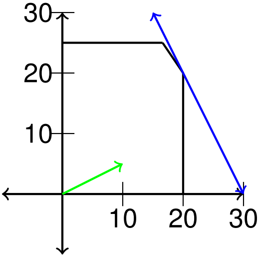
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor.

Again: simplex



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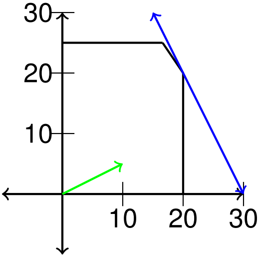
Simplex: Start at vertex. Move to better neighboring vertex. Until no better neighbor. Duality:

Again: simplex



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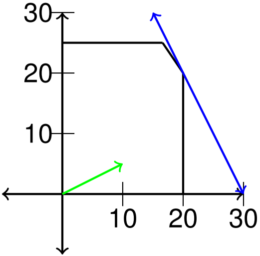
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?

Again: simplex



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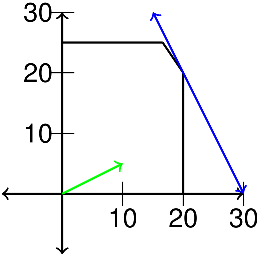
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.

Again: simplex



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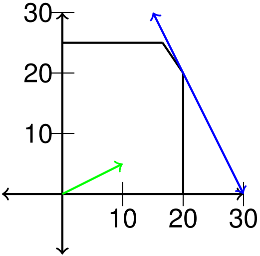
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$.

Again: simplex



$$\max 4x_1 + 2x_2$$
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$$3x_1 + 2x_2 \leq 100$$
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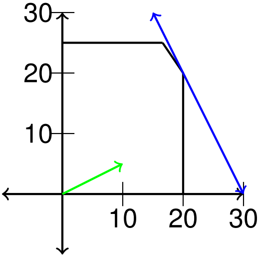
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!

Again: simplex



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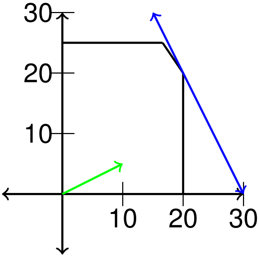
Again: simplex



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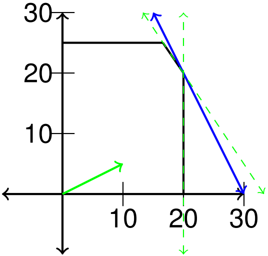
Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Geometrically and Complementary slackness:

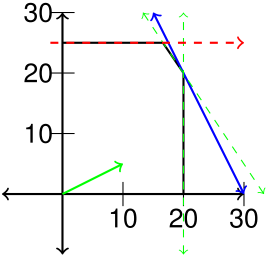
Again: simplex



$$\max 4x_1 + 2x_2$$
$$3x_1 \leq 60$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Geometrically and Complementary slackness:
Add tight constraints to "dominate objective function."

Again: simplex



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Don’t add this equation!

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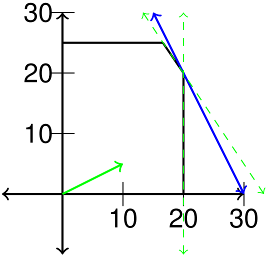

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$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Geometrically and Complementary slackness:
Add tight constraints to "dominate objective function."
Don't add this equation! Shifts.

Example: review.

$$\max x_1 + 8x_2$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1 + 2x_2 \leq 7$$
$$y_1, y_2, y_3 \geq 0$$

$$\min 4y_1 + 3y_2 + 7y_3$$
$$y_1 + y_3 \geq 1$$
$$y_2 + 2y_3 \geq 8$$
$$x_1, x_2 \geq 0$$

"Matrix form"

$$\max[1,8] \cdot [x_1, x_2]$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$
$$[x_1, x_2] \geq 0$$

$$\min[4,3,7] \cdot [y_1, y_2, y_3]$$
$$[y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$
$$[y_1, y_2, y_3] \geq 0$$

Matrix equations.

$$\max[1,8] \cdot [x_1, x_2] \qquad \min[4,3,7] \cdot [y_1, y_2, y_3]$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \qquad [y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$[x_1, x_2] \geq 0 \qquad [y_1, y_2, y_3] \geq 0$$

We can rewrite the above in matrix form.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \qquad c = [1,8] b = [4,3,7]$$

The primal is $Ax \leq b, \max c \cdot x, x \geq 0$.
The dual is $y^T A \geq c, \min b \cdot y, y \geq 0$.

See you on Tuesday.