Today

Routing and Experts.

Review: linear programming.

Taking Duals.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max congestion on any edge.

Two person game.

Row for every roting. ($A[r, e]$)

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e, r]$ - congestion of edge e on routing r.

m rows. Exponential number of columns.

Multiplicative Weights only maintains m weights.

Adversary only needs to provide best column each day.

Runtime only dependent on m and T (number of days.)

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1 - \varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights:
   $w_i = w_i(1 + \varepsilon)^{g_i/k}$.
2. Route all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T} \sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:
$G \geq G^*(1 - \varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon T}$
$T = \frac{k \log n}{\varepsilon^2} \rightarrow G^* - G \leq 2\varepsilon k$

$G^* = c_{\max}$ — Best row payoff against average routing.

$G \leq C$ — each day, gain is average congestion $\leq$ opt max congestion.
$\rightarrow c_{max} - C^* \leq 2\varepsilon k$ □

Better setup.

Runtime: $O(km)$ to route in each step.
$O(k \log n (\frac{1}{\varepsilon^2}))$ steps
$\rightarrow O(k^2 m \log n)$ to get a constant approximation.

Homework: $O(km \log n)$ algorithm.

Fractional versus Integer.

Did we solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1 + \varepsilon)$ optimal!

Homework 2. Problem 1.

Decent solution to path routing problem?

Randomized Rounding

For each s_i, t_i, choose path p_i with probability $f(p_i)$.

Congestion $c(e)$ edge rounds to $\tilde{c}(e)$.

Edge e.
used by paths $p_1, \ldots, p_m$.
Let $X_i = 1$,
if path p_i is chosen.
otherwise, $X_i = 0$.

Rounded congestion, $\tilde{c}(e)$, is $\sum_i X_i$.

Expected Congestion: $\sum_i E(X_i)$.
$E(X_i) = 1Pr[X_i = 1] + 0Pr[X_i = 0] = f(p_i)$
$\rightarrow \sum_i E(X_i) = \sum_i f(p_i) = c(e)$.
$\rightarrow E(\tilde{c}(e)) = c(e)$.

Concentration (law of large numbers)
$c(e)$ is relatively large ($\Omega(\log n)$)
$\rightarrow \tilde{c}(e) \approx c(e)$.

Concentration results? later.

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

Optimal point?

Try every point if we only had time!

How many points?

Real numbers?

Infinite. Uncountably infinite!

Profit maximization.

Plant Carrots or Peas?

2$ bushel of carrots. 4$ for peas.

Carrots take 3 unit of water/bushel.
Peas take 2 units of water/bushel.

100 units of water.

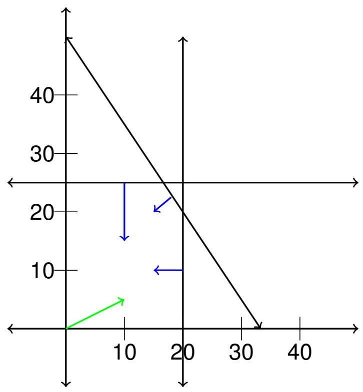
Peas require 2 yards/bushel of sunny land.
Carrots require 1 yard/bushel of shadyland.

Garden has 60 yards of sunny land and 75 yards of shady land.

To pea or not to pea, that is the question!

A linear program.

$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$



Optimal point?

To pea or not to pea.

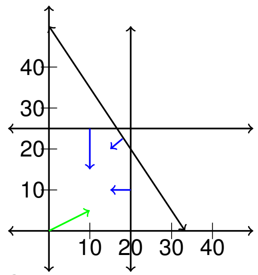
4$ for peas. 2$ bushel of carrots.
x_1- to pea! x_2 to carrot ?
Money $4x_1 + 2x_2$ maximize $\max 4x_1 + 2x_2$.
Carrots take 2 unit of water/bushel.
Peas take 3 units of water/bushel. 100 units of water.

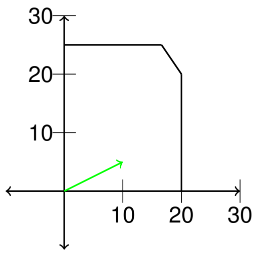
$3x_1 + 2x_2 \leq 100$

Peas 2 yards/bushel of sunny land. Have 40 sq yards.
$2x_1 \leq 40$
Carrots get 3 yards/bushel of shady land. Have 75 sq. yards.
$3x_2 \leq 75$
Can't make negative! $x_1, x_2 \geq 0$.
A linear program.

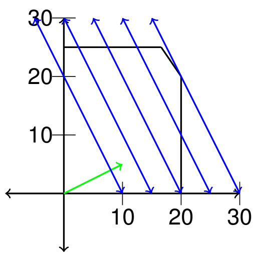
$$\begin{aligned} \max 4x_1 + 2x_2 \\ 2x_1 \leq 40 \\ 3x_2 \leq 75 \\ 3x_1 + 2x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

Feasible Region.

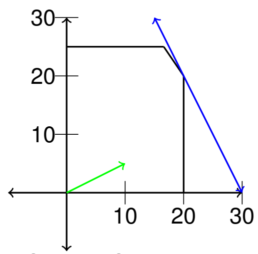




Convex.
Any two points in region connected by a line in region.
Algebraically:
If x and x' satisfy onstraint,
$\rightarrow x'' = \alpha x + (1-\alpha)x$



Optimal at pointy part of feasible region!
Vertex of region.
Intersection of two of the constraints (lines in two dimensions)!
Try every vertex! Choose best.
$O(m^2)$ if m constraints and 2 variables.
For n variables, m constraints, how many?
nm? $\binom{m}{n}$? $n+m$?
$\binom{m}{n}$
Finite!!!!!!
Exponential in the number of variables.



$$\max 4x_1 + 2x_2$$
$$3x_1 \leq 60$$
$$3x_2 \leq 75$$
$$3x_1 + 2x_2 \leq 100$$
$$x_1, x_2 \geq 0$$

Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. This example.
(0,0) objective 0. $\rightarrow (0,25)$ objective 50.
$\rightarrow (16\frac{2}{3}, 25)$ objective $115\frac{2}{3}$
$\rightarrow (20,20)$ objective 120.
Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Objective value: 120.
Can we do better? No!
Dual problem: add equations to get best upper bound.

Duality.

$$\max x_1 + 8x_2$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0$$

One Solution: $x_1 = 1, x_2 = 3$. Value is 25.

Best possible?

For any solution.
$x_1 \leq 4$ and $x_2 \leq 3$ ..
....so $x_1 + 8x_2 \leq 4 + 8(3) = 28$.
Added equation 1 and 8 times equation 2
yields bound on objective..
Better solution?
Better upper bound?

Duality.

$$\max x_1 + 8x_2$$
$$x_1 \leq 4$$
$$x_2 \leq 3$$
$$x_1 + 2x_2 \leq 7$$
$$x_1, x_2 \geq 0$$

Solution value: 25.
Add equation 1 and 8 times equation 2 gives..
$x_1 + 8x_2 \leq 4 + 24 = 28$.

Better way to add equations to get bound on function?
Sure: 6 times equation 2 and 1 times equation 3.
$x_1 + 8x_2 \leq 6(3) + 7 = 25$.

Thus, the value is at most 25.

The upper bound is same as solution!

Proof of optimality!

Duality:example

Idea: Add up positive linear combination of inequalities to "get" upper bound on optimization function.

Will this always work?

How to find best upper bound?

Duality: computing upper bound.

Best Upper Bound.

Multiplier	Inequality
y_1	$x_1 \leq 4$
y_2	$x_2 \leq 3$
y_3	$x_1 + 2x_2 \leq 7$

Adding equations thusly...
$$(y_1 + y_3)x_1 + (y_2 + 2y_3)x_2 \leq 4y_1 + 3y_2 + 7y_3.$$
The left hand side should "dominate" optimization function:

If $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

Find best y_i's to minimize upper bound?

The dual, the dual, the dual.

Find best y_i's to minimize upper bound?

Again: If you find $y_1, y_2, y_3 \geq 0$
and $y_1 + y_3 \geq 1$ and $y_2 + 2y_3 \geq 8$ then..
$x_1 + 8x_2 \leq 4y_1 + 3y_2 + 7y_3$

$$\begin{aligned} \min 4y_1 + 3y_2 + 7y_3 \\ y_1 + y_3 \geq 1 \\ y_2 + 2y_3 \geq 8 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

A linear program.
The Dual linear program.

Primal: $(x_1, x_2) = (1,3)$; Dual: $(y_1, y_2, y_3) = (0,6,1)$.

Value of both is 25!

Primal is optimal ... and dual is optimal!

The dual.

In general.

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Theorem: If a linear program has a bounded value, then its dual is bounded and has the same value.

Weak Duality: primal $(P) \leq$ dual (D)

Feasible (x, y)
$P(x) = c \cdot x \leq y^T Ax \leq y^T b \cdot x = D(y)$.

Strong Duality: next lecture.

Complementary Slackness

Primal LP	Dual LP
$\max c \cdot x$	$\min y^T b$
$Ax \leq b$	$y^T A \geq c$
$x \geq 0$	$y \geq 0$

Given A, b, c, and feasible solutions x and y.

Solutions x and y are both optimal if and only if
$x_i(c_i - (y^T A)_i) = 0$, and $y_j(b_j - (Ax)_j)$.

$x_i(c_i - (y^T A)_i) = 0 \rightarrow$
$\sum_i (c_i - (y^T A)_i) x_i = cx - y^T Ax \rightarrow cx = y^T Ax$.

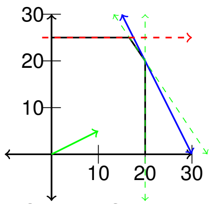
$y_j(b_j - (Ax)_j) = 0 \rightarrow$
$\sum_i y_j(b_j - (Ax)_j) = yb - y^T Ax \rightarrow by = y^T Ax$.

$cx = by$.

If both are feasible, $cx \leq by$, so must be optimal.

In words: nonzero dual variables only for tight constraints!

Again: simplex



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Simplex: Start at vertex. Move to better neighboring vertex.
Until no better neighbor. Duality:
Add blue equations to get objective function?
$1/3$ times first plus second.
Get $4x_1 + 2x_2 \leq 120$. Every solution must satisfy this inequality!
Geometrically and Complementary slackness:
Add tight constraints to "dominate objective function."
Don't add this equation! Shifts.

Example: review.

$\max x_1 + 8x_2$	$\min 4y_1 + 3y_2 + 7y_3$
$x_1 \leq 4$	$y_1 + y_3 \geq 1$
$x_2 \leq 3$	$y_2 + 2y_3 \geq 8$
$x_1 + 2x_2 \leq 7$	$x_1, x_2 \geq 0$
$y_1, y_2, y_3 \geq 0$	

"Matrix form"

$\max [1,8] \cdot [x_1, x_2]$ $\qquad$ $\min [4,3,7] \cdot [y_1, y_2, y_3]$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \qquad [y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$[x_1, x_2] \geq 0$ $\qquad$ $[y_1, y_2, y_3] \geq 0$

Matrix equations.

$\max [1,8] \cdot [x_1, x_2]$ $\qquad$ $\min [4,3,7] \cdot [y_1, y_2, y_3]$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \qquad [y_1, y_2, y_3] \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \geq \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$[x_1, x_2] \geq 0$ $\qquad$ $[y_1, y_2, y_3] \geq 0$

We can rewrite the above in matrix form.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \qquad c = [1,8] b = [4,3,7]$$

The primal is $Ax \leq b, \max c \cdot x, x \geq 0$.
The dual is $y^T A \geq c, \min b \cdot y, y \geq 0$.

See you on Tuesday.