Today.

Quick Review:
Quick Review:
experts framework/multiplicative weights algorithm
Quick Review:
   experts framework/multiplicative weights algorithm
Finish:
Today.

Quick Review:
- experts framework/multiplicative weights algorithm
Finish:
- randomized multiplicative weights algorithm for experts framework.
Quick Review:
   experts framework/multiplicative weights algorithm
Finish:
   randomized multiplicative weights algorithm for experts framework.
Equilibrium for two person games:
   using experts framework/MW algorithm.
Notes.

Got to definition of Approximate Equilibrium for zero sum games.
The multiplicative weights framework.
Expert’s framework.

$n$ experts.
Expert’s framework.

$n$ experts.

Every day, each offers a prediction.
$n$ experts.

Every day, each offers a prediction.

“Rain” or “Shine.”
Expert’s framework.

$n$ experts.

Every day, each offers a prediction.

“Rain” or “Shine.”

Whose advise do you follow?
Every day, each offers a prediction.

“The one who is correct most often.”

$n$ experts.

Whose advise do you follow?

“The one who is correct most often.”
Expert’s framework.

\[ n \] experts.
Every day, each offers a prediction.
“Rain” or “Shine.”
Whose advise do you follow?
“The one who is correct most often.”
Sort of.
Expert’s framework.

$n$ experts.
Every day, each offers a prediction.
“Rain” or “Shine.”
Whose advise do you follow?
“The one who is correct most often.”
Sort of.
How well do you do?
Infallible expert.

One of the expert’s is infallible!
Infallible expert.

One of the expert’s is infallible!

Your strategy?
Infallible expert.

One of the expert’s is infallible!

Your strategy?

Choose any expert that has not made a mistake!
Infallible expert.
One of the expert’s is infallible!
Your strategy?
Choose any expert that has not made a mistake!
How long to find perfect expert?
Infallible expert.

One of the expert’s is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..
Infallible expert.

One of the experts is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never!
Infallible expert.

One of the expert's is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.
Infallible expert.

One of the expert's is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?
Infallible expert.

One of the expert’s is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make?
Infallible expert.
One of the expert’s is infallible!
Your strategy?
Choose any expert that has not made a mistake!
How long to find perfect expert?
Maybe..never! Never see a mistake.
Better model?
How many mistakes could you make? Mistake Bound.
Infallible expert.

One of the expert’s is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? **Mistake Bound.**

(A) 1

(B) 2

(C) \( \log n \)

(D) \( n - 1 \)

Adversary designs setup to watch who you choose, and make that expert make a mistake.
Infallible expert.

One of the expert’s is infallible!

Your strategy?

Choose any expert that has not made a mistake!

How long to find perfect expert?

Maybe..never! Never see a mistake.

Better model?

How many mistakes could you make? Mistake Bound.

(A) 1
(B) 2
(C) \( \log n \)
(D) \( n - 1 \)

Adversary designs setup to watch who you choose, and make that expert make a mistake.

\( n - 1! \)
Concept Alert.

Note.
Concept Alert.

Note.

Adversary:
Concept Alert.

Note.

Adversary: makes you want to look bad.
Concept Alert.

Note.

Adversary:
  makes you want to look bad.
  "You could have done so well"...
Concept Alert.

Note.

Adversary:
- makes you want to look bad.
- "You could have done so well"
- but you didn’t!
Concept Alert.

Note.

Adversary:
  makes you want to look bad.
  "You could have done so well"...
  but you didn’t! ha..
Concept Alert.

Note.

Adversary:
- makes you want to look bad.
  "You could have done so well"...
  but you didn’t! ha..ha!
Note.

Adversary:
  makes you want to look bad.
  "You could have done so well"
  but you didn’t! ha..ha!

Analysis of Algorithms: do as well as possible!
Back to mistake bound.

Infallible Experts.
Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.
Infallible Experts.
Alg: Choose one of the perfect experts.
Mistake Bound: \( n - 1 \)
Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n - 1$

Lower bound: adversary argument.
Back to mistake bound.

Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n - 1$
  - Lower bound: adversary argument.
  - Upper bound:
Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n - 1$

- Lower bound: adversary argument.
- Upper bound: every mistake finds fallible expert.
Infallible Experts.

Alg: Choose one of the perfect experts.

**Mistake Bound: \( n - 1 \)**

- Lower bound: adversary argument.
- Upper bound: every mistake finds fallible expert.

Better Algorithm?
Back to mistake bound.

Infallible Experts.
Alg: Choose one of the perfect experts.

**Mistake Bound:** $n - 1$
  - Lower bound: adversary argument.
  - Upper bound: every mistake finds fallible expert.

Better Algorithm?
Making decision, not trying to find expert!
Infallible Experts.

Alg: Choose one of the perfect experts.

Mistake Bound: $n - 1$
- Lower bound: adversary argument.
- Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.
Infallible Experts.

Alg: Choose one of the perfect experts.

**Mistake Bound:** \( n - 1 \)
- Lower bound: adversary argument.
- Upper bound: every mistake finds fallible expert.

Better Algorithm?

Making decision, not trying to find expert!

Algorithm: Go with the majority of previously correct experts.

What you would do anyway!
Alg 2: find majority of the perfect experts

How many mistakes could you make?
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) $\log n$
(D) $n - 1$
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) \( \log n \)
(D) \( n - 1 \)

At most \( \log n \)!
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) $\log n$
(D) $n - 1$

At most $\log n$!

When alg makes a mistake, 
|“perfect” experts| drops by a factor of two.
Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1
(B) 2
(C) \(\log n\)
(D) \(n - 1\)

At most \(\log n\)!

When alg makes a mistake, “perfect” experts drops by a factor of two.

Initially \(n\) perfect experts
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) \( \log n \)
(D) \( n - 1 \)

At most \( \log n \)!

When alg makes a mistake, \(|\text{“perfect” experts}| \) drops by a factor of two.

Initially \( n \) perfect experts \( \Rightarrow \leq n/2 \) perfect experts
Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1
(B) 2
(C) $\log n$
(D) $n - 1$

At most $\log n$!

When alg makes a mistake, |“perfect” experts| drops by a factor of two.

Initially $n$ perfect experts mistake $\rightarrow \leq n/2$ perfect experts
mistake $\rightarrow \leq n/4$ perfect experts
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) \( \log n \)
(D) \( n - 1 \)

At most \( \log n \)!

When alg makes a mistake, |“perfect” experts| drops by a factor of two.

Initially \( n \) perfect experts mistake \( \rightarrow \) \( \leq \frac{n}{2} \) perfect experts
mistake \( \rightarrow \) \( \leq \frac{n}{4} \) perfect experts

\( \vdots \)
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) $\log n$
(D) $n - 1$

At most $\log n$!

When alg makes a mistake, $|\text{“perfect” experts}|$ drops by a factor of two.
Initially $n$ perfect experts mistake $\rightarrow \leq n/2$ perfect experts mistake $\rightarrow \leq n/4$ perfect experts

$\therefore$ mistake $\rightarrow \leq 1$ perfect expert
Alg 2: find majority of the perfect

How many mistakes could you make?

(A) 1
(B) 2
(C) $\log n$
(D) $n - 1$

At most $\log n$!

When alg makes a mistake,
|“perfect” experts| drops by a factor of two.

Initially $n$ perfect experts
mistake $\rightarrow \leq n/2$ perfect experts
mistake $\rightarrow \leq n/4$ perfect experts

$\vdots$
mistake $\rightarrow \leq 1$ perfect expert
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) log \( n \)
(D) \( n - 1 \)

At most \( \log n! \)

When alg makes a mistake,

|“perfect” experts| drops by a factor of two.

Initially \( n \) perfect experts

\begin{align*}
\text{mistake} &\rightarrow \leq n/2 \text{ perfect experts} \\
\text{mistake} &\rightarrow \leq n/4 \text{ perfect experts} \\
\vdots \\
\text{mistake} &\rightarrow \leq 1 \text{ perfect expert} \\
\geq 1 \text{ perfect expert}
\end{align*}
Alg 2: find majority of the perfect

How many mistakes could you make?
(A) 1
(B) 2
(C) \( \log n \)
(D) \( n - 1 \)

At most \( \log n \)!

When alg makes a mistake, “perfect” experts drops by a factor of two.

Initially \( n \) perfect experts mistake \( \rightarrow \) \( \leq n/2 \) perfect experts
mistake \( \rightarrow \) \( \leq n/4 \) perfect experts

\vdots

mistake \( \rightarrow \) \( \leq 1 \) perfect expert

\( \geq 1 \) perfect expert \( \rightarrow \) at most \( \log n \) mistakes!
Imperfect Experts

Goal?
Imperfect Experts

Goal?
Do as well as the best expert!
Imperfect Experts

Goal?
Do as well as the best expert!
Algorithm.
Imperfect Experts

Goal?
Do as well as the best expert!
Algorithm. Suggestions?
Imperfect Experts

Goal?
Do as well as the best expert!

Algorithm. Suggestions?
Go with majority?
Imperfect Experts

Goal?
Do as well as the best expert!
Algorithm. Suggestions?
Go with majority?
Penalize inaccurate experts?
Imperfect Experts

Goal?
Do as well as the best expert!
Algorithm. Suggestions?
Go with majority?
Penalize inaccurate experts?
Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i / 2$ if wrong.
Imperfect Experts

Goal?
Do as well as the best expert!
Algorithm. Suggestions?
Go with majority?
Penalize inaccurate experts?
Best expert is penalized the least.

1. Initially: $w_i = 1$. 


Imperfect Experts

Goal?
Do as well as the best expert!

Algorithm. Suggestions?
Go with majority?
Penalize inaccurate experts?
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Imperfect Experts

Goal?
Do as well as the best expert!

Algorithm. Suggestions?
Go with majority?
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Best expert is penalized the least.

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Imperfect Experts

Goal?
Do as well as the best expert!
Algorithm. Suggestions?
Go with majority?
Penalize inaccurate experts?
Best expert is penalized the least.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

1. Initially:
   \[ w_i = 1 \]

2. Predict with weighted majority of experts.

3. \[ w_i \rightarrow w_i / 2 \] if wrong.

Goal: Best expert makes \( m \) mistakes.

Potential function:

\[ \sum w_i \]

Initially \( n \).
For best expert, \( b \), \[ w_b \geq \frac{1}{2^m} \].

Each mistake:
- total weight of incorrect experts reduced by \(-\frac{1}{2}\) factor of \( 1/2 \).
- each incorrect expert weight multiplied by \( 1/2 \)!
- total weight decreases by factor of \( 3/4 \).

Mistake \( \rightarrow \) potential function decreased by \( 3/4 \).

We have \( \frac{1}{2} m \leq \sum w_i \leq (\frac{3}{4})^M n \).

where \( M \) is number of algorithm mistakes.
Analysis: weighted majority

1. Initially: \( w_i = 1 \).
2. Predict with weighted majority of experts.
3. \( w_i \rightarrow w_i/2 \) if wrong.

Goal: Best expert makes \( m \) mistakes.

Potential function:

\[
\sum_i w_i
\]

Initially \( n \). For best expert, \( b \), \( w_b \geq 1/2 m \).

Each mistake: total weight of incorrect experts reduced by \(-1/2\) factor of \(1/2\) each incorrect expert weight multiplied by \(1/2\) total weight decreases by factor of \(3/4\) mistake \(\rightarrow\) potential function decreased by \(3/4\).

We have \(1/2 m \leq \sum_i w_i \leq (3/4)^n M n\).
Analysis: weighted majority

Goal: Best expert makes \( m \) mistakes.

1. Initially: \( w_i = 1 \).
2. Predict with weighted majority of experts.
3. \( w_i \rightarrow w_i/2 \) if wrong.

Potential function:

\[
\sum_i w_i
\]

Initially \( n \).

For best expert, \( w_b \geq 1/2 \).

Each mistake: total weight of incorrect experts reduced by \( -\frac{1}{2} \) factor of \( \frac{1}{2} \) each incorrect expert weight multiplied by \( \frac{1}{2} \) total weight decreases by \( \frac{3}{4} \) factor of \( \frac{3}{4} \), mistake \( \geq \) half weight with incorrect experts. Mistake \( \rightarrow \) potential function decreased by \( \frac{3}{4} \).

We have \( \frac{1}{2} m \leq \sum_i w_i \leq \left( \frac{3}{4} \right) M n \).

where \( M \) is number of algorithm mistakes.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function:

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Mistake $\rightarrow$ potential function decreased by $\frac{3}{4}$.

We have $\frac{1}{2}m \leq \sum w_i \leq \left(\frac{3}{4}\right)^M n$, where $M$ is the number of algorithm mistakes.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

Potential function: $\sum_i w_i$.

For best expert, $b$, $w_b \geq 1/2 m$.

Each mistake: total weight of incorrect experts reduced by $-\frac{1}{2}$.

Total weight decreases by $\frac{3}{4}$ factor of each incorrect expert weight multiplied by $1/2$.

Potential function decreased by $\frac{3}{4}$.

We have $\frac{1}{2} m \leq \sum_i w_i \leq \left(\frac{3}{4}\right) M n$, where $M$ is number of algorithm mistakes.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.
Potential function: $\sum w_i$. Initially $n$.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

1. Initially: $w_i = 1$.

2. Predict with weighted majority of experts.

3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes \( m \) mistakes.

Potential function: \( \sum_i w_i \). Initially \( n \).

For best expert, \( b \), \( w_b \geq \frac{1}{2^m} \).

Each mistake:

1. Initially: \( w_i = 1 \).
2. Predict with weighted majority of experts.
3. \( w_i \rightarrow w_i/2 \) if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake:
- total weight of incorrect experts reduced by

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.

We have $\frac{1}{2} m \leq \sum_i w_i \leq (\frac{3}{4})^m M n$.

where $M$ is number of algorithm mistakes.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake: 
- total weight of incorrect experts reduced by $-\frac{1}{2}$?

1. Initially: $w_i = 1$.

2. Predict with weighted majority of experts.

3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake:
- total weight of incorrect experts reduced by $-1$? $-2$?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake:
  total weight of incorrect experts reduced by $-1? -2? \text{ factor of } \frac{1}{2}?$

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes \( m \) mistakes.

Potential function: \( \sum_i w_i \). Initially \( n \).

For best expert, \( b \), \( w_b \geq \frac{1}{2^m} \).

Each mistake:
- total weight of incorrect experts reduced by \(-1^? - 2^? \) factor of \( \frac{1}{2}^? \)
- each incorrect expert weight multiplied by \( \frac{1}{2}! \)

1. Initially: \( w_i = 1 \).
2. Predict with weighted majority of experts.
3. \( w_i \rightarrow w_i/2 \) if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake:
- total weight of incorrect experts reduced by $-1/2$?
- each incorrect expert weight multiplied by $1/2$!
- total weight decreases by

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2m}$.

Each mistake:
- total weight of incorrect experts reduced by $-1\frac{1}{2}$?
- each incorrect expert weight multiplied by $\frac{1}{2}$?
- total weight decreases by factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake:
- total weight of incorrect experts reduced by $-1/2^? - 2/2^?$ factor of $1/2^$?
- each incorrect expert weight multiplied by $1/2^$!
- total weight decreases by factor of $1/2^$? factor of $3/4^$?
- mistake $\rightarrow \geq$ half weight with incorrect experts.

1. Initially: $w_i = 1$.

2. Predict with weighted majority of experts.

3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake:
- total weight of incorrect experts reduced by $-1$? $-2$? factor of $\frac{1}{2}$?
- each incorrect expert weight multiplied by $\frac{1}{2}$!
- total weight decreases by factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?
- mistake $\rightarrow \geq$ half weight with incorrect experts.

Mistake $\rightarrow$ potential function decreased by $\frac{3}{4}$.

1. Initially: $w_i = 1$.
2. Predict with weighted majority of experts.
3. $w_i \rightarrow w_i/2$ if wrong.
Analysis: weighted majority

Goal: Best expert makes $m$ mistakes.

Potential function: $\sum_i w_i$. Initially $n$.

For best expert, $b$, $w_b \geq \frac{1}{2^m}$.

Each mistake:
- total weight of incorrect experts reduced by $-1$?
- $-2$? factor of $\frac{1}{2}$?
- each incorrect expert weight multiplied by $\frac{1}{2}$!
- total weight decreases by factor of $\frac{1}{2}$? factor of $\frac{3}{4}$?
- mistake $\rightarrow \geq$ half weight with incorrect experts.

Mistake $\rightarrow$ potential function decreased by $\frac{3}{4}$.

We have

$$\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.$$
Analysis: continued.

\[
\frac{1}{2^m} \leq \sum_i w_i \leq \left( \frac{3}{4} \right)^M n.
\]
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n. \]

\(m\) - best expert mistakes
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n. \]

\( m \) - best expert mistakes \hspace{1cm} \( M \) algorithm mistakes.
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq \left( \frac{3}{4} \right)^M n. \]

- \( m \) - best expert mistakes
- \( M \) - algorithm mistakes.

\[ \frac{1}{2^m} \leq \left( \frac{3}{4} \right)^M n. \]
Analysis: continued.

\[
\frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n.
\]

\( m \) - best expert mistakes \( M \) algorithm mistakes.

\[
\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.
\]

Take log of both sides.

\[
\frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n.
\]

\( M \) \leq \frac{(m + \log n)}{\log \left(\frac{4}{3}\right)} \leq 2\left(1 + \epsilon\right)m + 2\ln n \epsilon
\]

Approaches a factor of two of best expert performance!
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq \left( \frac{3}{4} \right)^M n. \]

\( m \) - best expert mistakes \( M \) algorithm mistakes.

\[ \frac{1}{2^m} \leq \left( \frac{3}{4} \right)^M n. \]

Take log of both sides.

\[ -m \leq -M \log(4/3) + \log n. \]
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq \left( \frac{3}{4} \right)^M n. \]

$m$ - best expert mistakes \hspace{1cm} $M$ algorithm mistakes.

\[ \frac{1}{2^m} \leq \left( \frac{3}{4} \right)^M n. \]

Take log of both sides.

\[ -m \leq -M \log(4/3) + \log n. \]

Solve for $M$.

\[ M \leq (m + \log n) / \log(4/3) \]
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq (\frac{3}{4})^M n. \]

\( m \) - best expert mistakes \( M \) algorithm mistakes.

\[ \frac{1}{2^m} \leq (\frac{3}{4})^M n. \]

Take log of both sides.

\[ -m \leq -M \log(4/3) + \log n. \]

Solve for \( M \).

\[ M \leq (m + \log n)/\log(4/3) \leq 2.4(m + \log n) \]
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq \left( \frac{3}{4} \right)^M n. \]

- $m$ - best expert mistakes
- $M$ - algorithm mistakes.

\[ \frac{1}{2^m} \leq \left( \frac{3}{4} \right)^M n. \]

Take log of both sides.

\[ -m \leq -M \log(4/3) + \log n. \]

Solve for $M$.

\[ M \leq (m + \log n) / \log(4/3) \leq 2.4(m + \log n) \]

Multiple by $1 - \varepsilon$ for incorrect experts...
Analysis: continued.

\[ \frac{1}{2^m} \leq \sum_i w_i \leq \left(\frac{3}{4}\right)^M n. \]

- \( m \) - best expert mistakes
- \( M \) - algorithm mistakes.

\[ \frac{1}{2^m} \leq \left(\frac{3}{4}\right)^M n. \]

Take log of both sides.

\[ -m \leq -M \log(4/3) + \log n. \]

Solve for \( M \).

\[ M \leq (m + \log n) / \log(4/3) \leq 2.4(m + \log n) \]

Multiple by \( 1 - \epsilon \) for incorrect experts...

\[ (1 - \epsilon)^m \leq \left(1 - \frac{\epsilon}{2}\right)^M n. \]
Analysis: continued.

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\[ M \leq 2(1 + \varepsilon)m + \frac{2\ln n}{\varepsilon} \]
Analysis: continued.

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\[ m \text{ - best expert mistakes} \quad M \text{ algorithm mistakes.} \]

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Approaches a factor of two of best expert performance!
Best Analysis?

Two experts: A, B

Which is worse?

(A) A right on even, B right on odd.
(B) A right first half of days, B right second half of days.

Best expert performance: $T/2$ mistakes.

Pattern (A): $T - 1$ mistakes.

Factor of almost two worse!
Best Analysis?

Two experts: A, B

Bad example?
Best Analysis?

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Bad example?

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Factor of (almost) two worse!
Randomization

Better approach?
Randomization

Better approach?
Use?
Randomization!!!!

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Randomization!!!!

Better approach?
Use?
   Randomization!
That is, choose expert $i$ with prob $\propto w_i$
Better approach?
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Bad example: A,B,A,B,A...
Better approach?
Use?
  Randomization!
That is, choose expert $i$ with prob $\propto w_i$
Bad example: A,B,A,B,A...
After a bit, A and B make nearly the same number of mistakes.
Randomization!!!

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Bad example: A,B,A,B,A...

After a bit, A and B make nearly the same number of mistakes.
Choose each with approximately the same probability.
Randomization!!!!

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Bad example: A,B,A,B,A...

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Choose each with approximately the same probability.
Make a mistake around $1/2$ of the time.
Better approach?  
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Roughly
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Roughly optimal!
Randomized analysis.

Some formulas:
Randomized analysis.

Some formulas:
For $\varepsilon \leq 1$, $x \in [0, 1]$,
Randomized analysis.

Some formulas:
For $\epsilon \leq 1, x \in [0, 1],$

$$(1 + \epsilon)^x \leq (1 + \epsilon x)$$

$$(1 - \epsilon)^x \leq (1 - \epsilon x)$$
Randomized analysis.

Some formulas:

For \( \varepsilon \leq 1, x \in [0, 1] \),

\[
(1 + \varepsilon)^x \leq (1 + \varepsilon x)
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For \( \varepsilon \in [0, \frac{1}{2}] \),
Randomized analysis.

Some formulas:

For $\varepsilon \leq 1, x \in [0, 1]$,

$$(1 + \varepsilon)^x \leq (1 + \varepsilon x)$$
$$(1 - \varepsilon)^x \leq (1 - \varepsilon x)$$

For $\varepsilon \in [0, \frac{1}{2}]$,

$$-\varepsilon - \varepsilon^2 \leq \ln(1 - \varepsilon) \leq -\varepsilon$$
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Randomized analysis.

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$$\varepsilon - \varepsilon^2 \leq \ln(1 + \varepsilon) \leq \varepsilon$$

Proof Idea: $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$
Randomized algorithm

Losses in \([0, 1]\).
Randomized algorithm

Losses in $[0, 1]$. 

Expert $i$ loses $\ell_i^t \in [0, 1]$ in round $t$. 

Initially $w_i = 1$ for expert $i$. 

Choose expert $i$ with prob $w_i/W$, $W = \sum_i w_i$. 

$w_i \leftarrow w_i (1 - \epsilon) \ell_i^t W(t)$ sum of $w_i$ at time $t$. 

Best expert, $b$, loses $L^*$ total. 

$W(t) \geq w_b \geq (1 - \epsilon) L^*$. 

$L_t = \sum_i w_i \ell_i^t$ expected loss of alg. in time $t$. 

Claim: $W(t+1) \leq W(t) (1 - \epsilon L_t)$. 

Loss $\rightarrow$ weight loss. 

Proof: $W(t+1) \leq \sum_i (1 - \epsilon \ell_i^t) w_i \leq \sum_i w_i - \epsilon \sum_i w_i \ell_i^t = \sum_i w_i (1 - \epsilon \ell_i^t \sum_i w_i \ell_i^t)$ 

$= W(t) (1 - \epsilon L_t)$.
Randomized algorithm

Losses in $[0, 1]$.
Expert $i$ loses $\ell_i^t \in [0, 1]$ in round $t$.

1. Initially $w_i = 1$ for expert $i$. 
Randomized algorithm

Losses in $[0, 1]$.

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$W(t)$ sum of $w_i$ at time $t$. 

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$L_t = \sum_i \frac{w_i \ell_i^t}{W}$ expected loss of alg. in time $t$. 
Randomized algorithm

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Claim: $W(t + 1) \leq W(t)(1 - \varepsilon L_t)$
Randomized algorithm

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Claim: \(W(t + 1) \leq W(t)(1 - \epsilon L_t)\) Loss \(\rightarrow\) weight loss.

Proof:
Randomized algorithm

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$$W(t + 1) \leq \sum_i (1 - \varepsilon \ell^t_i) w_i$$
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$$= \sum_i w_i \left( 1 - \varepsilon L_t \right)$$
Analysis

\[(1 - \varepsilon)^{L^*} \leq W(T) \leq n \prod_t (1 - \varepsilon L_t)\]
Analysis

\[(1 - \varepsilon)^{L^*} \leq W(T) \leq n \ \prod_t (1 - \varepsilon L_t)\]

Take logs

\[(L^*) \ln(1 - \varepsilon) \leq \ln n + \sum \ln(1 - \varepsilon L_t)\]
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\[(L^*) \ln(1 - \varepsilon) \leq \ln n + \sum \ln(1 - \varepsilon L_t)\]

Use \(-\varepsilon - \varepsilon^2 \leq \ln(1 - \varepsilon) \leq -\varepsilon\)
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\[(L^*) \ln(1 - \varepsilon) \leq \ln n + \sum \ln(1 - \varepsilon L_t)\]

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\[-(L^*)(\varepsilon + \varepsilon^2) \leq \ln n - \varepsilon \sum L_t\]
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\[(L^*) \ln(1 - \varepsilon) \leq \ln n + \sum \ln(1 - \varepsilon L_t)\]

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And

\[
\sum_t L_t \leq (1 + \varepsilon) L^* + \ln n \varepsilon.
\]

Within \((1 + \varepsilon)\)ish of the best expert!

No factor of 2 loss!
Analysis

\[(1 - \varepsilon)^{L^*} \leq W(T) \leq n \prod_t (1 - \varepsilon L_t)\]

Take logs
\[(L^*) \ln(1 - \varepsilon) \leq \ln n + \sum \ln(1 - \varepsilon L_t)\]

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\[-(L^*)(\varepsilon + \varepsilon^2) \leq \ln n - \varepsilon \sum L_t\]

And
\[\sum_t L_t \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon} .\]
Analysis

\[(1 - \varepsilon)^{L^*} \leq W(T) \leq n \prod_t (1 - \varepsilon L_t)\]

Take logs
\[\left(L^*\right) \ln(1 - \varepsilon) \leq \ln n + \sum \ln(1 - \varepsilon L_t)\]

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\[\sum_t L_t \leq (1 + \varepsilon) L^* + \frac{\ln n}{\varepsilon}\]

\[\sum_t L_t\] is total expected loss of algorithm.
Analysis

\[(1 - \varepsilon)^{L^*} \leq W(T) \leq n \prod_t (1 - \varepsilon L_t)\]

Take logs
\[(L^*) \ln (1 - \varepsilon) \leq \ln n + \sum \ln (1 - \varepsilon L_t)\]

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Within \((1 + \varepsilon)\)
Analysis

$$(1 - \varepsilon)^{L^*} \leq W(T) \leq n \prod_t (1 - \varepsilon L_t)$$

Take logs

$$(L^*) \ln(1 - \varepsilon) \leq \ln n + \sum \ln(1 - \varepsilon L_t)$$

Use $-\varepsilon - \varepsilon^2 \leq \ln(1 - \varepsilon) \leq -\varepsilon$

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And

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Within $(1 + \varepsilon)$ ish
Analysis

\[(1 - \varepsilon)^{L^*} \leq W(T) \leq n \prod_t (1 - \varepsilon L_t)\]

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\(\sum_t L_t\) is total expected loss of algorithm.

Within \((1 + \varepsilon)\) ish of the best expert!
(1 − ε)^{L^*} \leq W(T) \leq n \prod_t (1 − εL_t)

Take logs
(L^*) \ln(1 − ε) \leq \ln n + \sum \ln(1 − εL_t)

Use −ε − ε^2 \leq \ln(1 − ε) \leq −ε

−(L^*)(ε + ε^2) \leq \ln n − ε \sum L_t

And

\sum_t L_t \leq (1 + ε)L^* + \frac{\ln n}{ε}.

\sum_t L_t \text{ is total expected loss of algorithm.}

Within (1 + ε) ish of the best expert!

No factor of 2 loss!
Gains.

Why so negative?
Gains.

Why so negative?
Each day, each expert gives gain in $[0, 1]$. 
Gains.

Why so negative?
Each day, each expert gives gain in $[0, 1]$.
Multiplicative weights with $(1 + \varepsilon)^{g_i^t}$.
Gains.

Why so negative?
Each day, each expert gives gain in $[0, 1]$.
Multiplicative weights with $(1 + \varepsilon)^{g_i^t}$.

$$G \geq (1 - \varepsilon)G^* - \frac{\log n}{\varepsilon}$$

where $G^*$ is payoff of best expert.
Why so negative?
Each day, each expert gives gain in $[0, 1]$.
Multiplicative weights with $(1 + \varepsilon)^{g^t_i}$.

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Scaling:
Gains.

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Multiplicative weights with $(1 + \varepsilon)^{g_i^t}$.

$$G \geq (1 - \varepsilon)G^* - \frac{\log n}{\varepsilon}$$

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Scaling:
Not $[0, 1]$, say $[0, \rho]$. 
Gains.

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Each day, each expert gives gain in $[0, 1]$.
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where $G^*$ is payoff of best expert.

Scaling:
Not $[0, 1]$, say $[0, \rho]$.

$$L \leq (1 + \varepsilon)L^* + \frac{\rho \log n}{\varepsilon}$$
Summary: multiplicative weights.
Summary: multiplicative weights.

Framework: $n$ experts, each loses different amount every day.

Perfect Expert: $\log n$ mistakes.
Imperfect Expert: best makes $m$ mistakes.
Deterministic Strategy: $2(1+\varepsilon)m + \log n \varepsilon$.
Real numbered losses: Best loses $L^*$ total.
Randomized Strategy: $(1+\varepsilon)L^* + \log n \varepsilon$.
Strategy: Choose proportional to weights multiply weight by $(1-\varepsilon)$ loss.

Multiplicative weights framework! Applications next!
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Multiplicative weights framework!
Applications next!
Two person zero sum games.  

$m \times n$ payoff matrix $A$. 

Row mixed strategy: 
\[ x = (x_1, \ldots, x_m) \]  
Column mixed strategy: 
\[ y = (y_1, \ldots, y_n) \]  
Payoff for strategy pair $(x, y)$: 
\[ p(x, y) = x^t A y \]  
That is, 
\[ \sum_i x_i \left( \sum_j a_{ij} y_j \right) = \sum_j \left( \sum_i x_i a_{ij} \right) y_j \]  
Recall row minimizes, column maximizes.  

Equilibrium pair:  
\[ (x^*, y^*) \]  
\[ (x^*)^t A y^* = \max_y \left( (x^*)^t A y \right) = \min_x x^t A y^* \]  
(No better column strategy, no better row strategy.)
Two person zero sum games.

$m \times n$ payoff matrix $A$.

Row mixed strategy: $x = (x_1, \ldots, x_m)$. 

Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$ s.t. $x^t Ay^* = \max_y (x^t Ay) = \min_x x^t Ay^*$. (No better column strategy, no better row strategy.)
Two person zero sum games.

$m \times n$ payoff matrix $A$.

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That is, $\sum_i x_i \left( \sum_j a_{ij} y_j \right) = \sum_j \left( \sum_i x_i a_{ij} \right) y_j$.

Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$(x^*)^T A y^* = \max_y (x^*)^T A y = \min_x x^T A y^*$. (No better column strategy, no better row strategy.)
Two person zero sum games.

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Row mixed strategy: $x = (x_1, \ldots, x_m)$.
Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair $(x, y)$:

$$p(x, y) = x^t Ay$$

That is,

$$\sum_i x_i \left( \sum_j a_{i,j} y_j \right) = \sum_j \left( \sum_i x_i a_{i,j} \right) y_j.$$
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Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?
Two person zero sum games.

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Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$$\left(x^*\right)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.$$ 

(No better column strategy, no better row strategy.)
Equilibrium.

Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*. \]

(No better column strategy, no better row strategy.)
Equilibrium.

Equilibrium pair: \((x^*, y^*)\)?

\[
p(x, y) = (x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.
\]

(No better column strategy, no better row strategy.)

No row is better:

\[
\min_i A^{(i)} \cdot y = (x^*)^t Ay^*. \quad 1
\]

\(^1 A^{(i)} \) is \(i\)th row.
Equilibrium.

Equilibrium pair: \((x^*, y^*)\)?

\[
p(x, y) = (x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.\]

(No better column strategy, no better row strategy.)

No row is better:

\[
\min_i A^{(i)} \cdot y = (x^*)^t Ay^*. \quad 1
\]

No column is better:

\[
\max_j (A^t)^{(j)} \cdot x = (x^*)^t Ay^*.\]
Best Response

Column goes first:

Find $y$, where best row is not too low.

$$R = \max_y \min_x (x \cdot Ay)$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots, 0)$.

Example: Roshambo.

Value of $R$?

Row goes first:

Find $x$, where best column is not high.

$$C = \min_x \max_y (x \cdot Ay)$$

Again: $y$ of form $(0, 0, \ldots, 1, \ldots, 0)$.

Example: Roshambo.

Value of $C$?
Best Response

**Column goes first:**
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t A y).$$

---

Note: $x$ can be $(0, 0, \ldots, 1, \ldots, 0)$.

Example: Roshambo.

Value of $R$?

**Row goes first:**
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Column goes first:
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Column goes first:
Find $y$, where best row is not too low.

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Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of $R$?
Best Response

**Column goes first:**
Find \( y \), where best row is not too low..

\[
R = \max_y \min_x (x^t A y).
\]

Note: \( x \) can be \((0,0,\ldots,1,\ldots0)\).

Example: Roshambo. Value of \( R \)?

**Row goes first:**
Find \( x \), where best column is not high.
Best Response

Column goes first:
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.
Example: Roshambo. Value of $R$?

Row goes first:
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$
Best Response

Column goes first:
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.
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Example: Roshambo.
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Example: Roshambo. Value of $R$?

Row goes first:
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$

Again: $y$ of form $(0, 0, \ldots, 1, \ldots 0)$.
Example: Roshambo. Value of $C$?
Duality.

\[ R = \max_{\substack{y \in Y}} \min_{\substack{x \in X}} (x^t Ay). \]
Duality.

\[ R = \max_{y} \min_{x} (x^t Ay). \]
\[ C = \min_{x} \max_{y} (x^t Ay). \]
Duality.

\[
R = \max_y \min_x (x^t Ay).
\]

\[
C = \min_x \max_y (x^t Ay).
\]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.
Duality.

\[ R = \max_y \min_x (x^t A y). \]
\[ C = \min_x \max_y (x^t A y). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \(v\):
row payoffs \((Ay^*)\) all \(\geq v \implies R \geq v.\)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
- row payoffs \((Ay^*)\) all \( \geq v \) \( \implies R \geq v. \)
- column payoffs \(((x^*)^t A)\) all \( \leq v \)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \(v\):
- row payoffs \((Ay^*)\) all \(\geq v \implies R \geq v.\)
- column payoffs \(((x^*)^t A)\) all \(\leq v \implies v \geq C.\)
Duality.

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\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):

row payoffs \((Ay^*)\) all \( \geq v \) \( \implies R \geq v. \)

column payoffs \(((x^*)^tA)\) all \( \leq v \) \( \implies v \geq C. \)

\[ \implies R \geq C \]
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \(v\):

- row payoffs \((Ay^*)\) all \( \geq v \) \(\implies\) \( R \geq v \).
- column payoffs \(((x^*)^t A)\) all \( \leq v \) \(\implies\) \( v \geq C \).

\[ \implies R \geq C \]

Equilibrium \(\implies R = C\)!
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)
**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \) \( \implies R \geq v. \)
column payoffs \(((x^*)^t A)\) all \( \leq v \) \( \implies v \geq C. \)
\[ \implies R \geq C \]

Equilibrium \( \implies R = C! \)

**Strong Duality:** There is an equilibrium point!
Duality.

\[
R = \max_y \min_x (x^t Ay).
\]

\[
C = \min_x \max_y (x^t Ay).
\]

**Weak Duality:** \( R \leq C \).

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \(\geq v \implies R \geq v \).
column payoffs \(((x^*)^t A)\) all \(\leq v \implies v \geq C \).
\[ \implies R \geq C \]

Equilibrium \(\implies R = C \! \! \)!

**Strong Duality:** There is an equilibrium point! and \( R = C \! \! \).
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \) \( \implies \) \( R \geq v. \)
column payoffs \(((x^*)^t A)\) all \( \leq v \) \( \implies \) \( v \geq C. \)
\( \implies R \geq C \)

Equilibrium \( \implies R = C! \)

**Strong Duality:** There is an equilibrium point! and \( R = C! \)

Doesn’t matter who plays first!
Proof of Equilibrium.

Later.
Proof of Equilibrium.

Later. Still later...
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

\[ C(x) = \max_y x^t Ay \]
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]

Always: \( R(y) \leq C(x) \)
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]

Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]

Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)

Equilibrium: \((x, y)\)
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

$C(x) = \max_y x^t Ay$

$R(y) = \min_x x^t Ay$

Always: $R(y) \leq C(x)$

Strategy pair: $(x, y)$

Equilibrium: $(x, y)$

$R(y) = C(x)$
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]

Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)

Equilibrium: \((x, y)\)

\[ R(y) = C(x) \rightarrow C(x) - R(y) = 0. \]
Proof of Equilibrium.

Later. Still later...

Aproximate equilibrium ...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]

Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)

Equilibrium: \((x, y)\)

\[ R(y) = C(x) \rightarrow C(x) - R(y) = 0. \]

Approximate Equilibrium: \( C(x) - R(y) \leq \varepsilon. \)
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]

Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)

Equilibrium: \((x, y)\)

\[ R(y) = C(x) \rightarrow C(x) - R(y) = 0. \]

Approximate Equilibrium: \( C(x) - R(y) \leq \varepsilon. \)

With \( R(y) \leq C(x) \)
Proof of Equilibrium.

Later. Still later...

Aproximate equilibrium ...

\[ C(x) = \max_y x^t Ay \]
\[ R(y) = \min_x x^t Ay \]
Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)

Equilibrium: \((x, y)\)

\[ R(y) = C(x) \rightarrow C(x) - R(y) = 0. \]

Approximate Equilibrium: \( C(x) - R(y) \leq \varepsilon. \)

With \( R(y) \leq C(x) \)
\[ \rightarrow \text{“Response } y \text{ to } x \text{ is within } \varepsilon \text{ of best response”} \]
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

\[ C(x) = \max_y x^t A y \]
\[ R(y) = \min_x x^t A y \]

Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)

Equilibrium: \((x, y)\)

\[ R(y) = C(x) \rightarrow C(x) - R(y) = 0. \]

Approximate Equilibrium: \( C(x) - R(y) \leq \epsilon. \)

With \( R(y) \leq C(x) \)

→ “Response \( y \) to \( x \) is within \( \epsilon \) of best response”

→ “Response \( x \) to \( y \) is within \( \epsilon \) of best response”
Proof of Equilibrium.

Later. Still later...

Approximate equilibrium ...

\[ C(x) = \max_y x^t A y \]
\[ R(y) = \min_x x^t A y \]

Always: \( R(y) \leq C(x) \)

Strategy pair: \((x, y)\)

Equilibrium: \((x, y)\)

\[ R(y) = C(x) \rightarrow C(x) - R(y) = 0. \]

Approximate Equilibrium: \( C(x) - R(y) \leq \varepsilon. \)

With \( R(y) \leq C(x) \)

\( \rightarrow \) “Response \( y \) to \( x \) is within \( \varepsilon \) of best response”

\( \rightarrow \) “Response \( x \) to \( y \) is within \( \varepsilon \) of best response”
Proof of approximate equilibrium.

How?

(A) Using geometry.
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
(C) Using multiplicative weights.
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
(C) Using multiplicative weights.
(D) By the skin of my teeth.
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
(C) Using multiplicative weights.
(D) By the skin of my teeth.

(C)
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
(C) Using multiplicative weights.
(D) By the skin of my teeth.

(C) ..and (D).
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
(C) Using multiplicative weights.
(D) By the skin of my teeth.

(C) ..and (D).
Not hard.
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
(C) Using multiplicative weights.
(D) By the skin of my teeth.

(C) ..and (D).
Not hard. Even easy.
Proof of approximate equilibrium.

How?

(A) Using geometry.
(B) Using a fixed point theorem.
(C) Using multiplicative weights.
(D) By the skin of my teeth.

(C) ..and (D).
Not hard. Even easy. Still, head scratching happens.
Games and experts

Again: find \((x^*, y^*)\), such that
Games and experts

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\[
(max_y x^* Ay) - (min_x x^* Ay^*) \leq \varepsilon
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\left( \max_y x^* Ay \right) - \left( \min_x x^* Ay^* \right) \leq \epsilon
\]

\[
C(x^*) - R(y^*) \leq \epsilon
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Games and experts

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Experts Framework: \(n\) Experts, \(T\) days, \(L^*\) -total loss.
Again: find \((x^*, y^*)\), such that

\[
(\max_y x^* Ay) - (\min_x x^* Ay^*) \leq \varepsilon
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Experts Framework: \(n\) Experts, \(T\) days, \(L^*\) -total loss.

Multiplicative Weights Method yields loss \(L\) where
Again: find \((x^*, y^*)\), such that

\[
\left( \max_y x^* Ay \right) - \left( \min_x x^* Ay^* \right) \leq \varepsilon
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C(x^*) - R(y^*) \leq \varepsilon
\]

Experts Framework: \(n\) Experts, \(T\) days, \(L^*\) -total loss.

Multiplicative Weights Method yields loss \(L\) where

\[
L \leq \left( 1 + \varepsilon \right) L^* + \frac{\log n}{\varepsilon}
\]
Assume: $A$ has payoffs in $[0, 1]$. For $T = \log n \in \mathbb{R}_{>0}$ days:

1) $m$ pure row strategies are experts. Use multiplicative weights, produce row distribution.

Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

2) Each day, adversary plays best column response to $x_t$. Choose column of $A$ that maximizes row's expected loss.

Let $y_t$ be indicator vector for this column. Let $y^* = 1_T \sum y_t$ and $x^* = \arg\min_x x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Proof Idea: $x_t$ minimizes the best column response is chosen. Clearly good for row. Column best response is at least what it is against $x_t$. Total loss, $L$ is at least column payoff. Best row payoff, $L^*$ is roughly less than $L$ due to MW analysis. Combine bounds. Done!
Games and Experts.

Assume: \( A \) has payoffs in \([0, 1]\).
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) $m$ pure row strategies are experts. Use multiplicative weights, produce row distribution.

Let $x_t$ be distribution (row strategy) on day $t$.

2) Each day, adversary plays best column response to $x_t$. Choose column of $A$ that maximizes row's expected loss. Let $y_t$ be indicator vector for this column.

Let $y^* = \frac{1}{T} \sum y_t$ and $x^* = \text{argmin} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Proof Idea: $x_t$ minimizes the best column response is chosen. Clearly good for row. Column best response is at least what it is against $x_t$.

Total loss, $L$ is at least $\text{column payoff}$. Best row payoff, $L^*$ is roughly less than $L$ due to MW analysis. Combine bounds. Done!
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) $m$ pure row strategies are experts.
Games and Experts.

Assume: A has payoffs in [0, 1].

For $T = \frac{\log n}{\epsilon^2}$ days:

1) $m$ pure row strategies are experts.

Use multiplicative weights, produce row distribution.
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) $m$ pure row strategies are experts.
Use multiplicative weights, produce row distribution.
Let $x_t$ be distribution (row strategy) $x_t$ on day $t$. 
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\epsilon^2}$ days:

1) $m$ pure row strategies are experts.
   Use multiplicative weights, produce row distribution.
   Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

2) Each day, adversary plays best column response to $x_t$.
   Choose column of $A$ that maximizes row's expected loss.
   Let $y_t$ be indicator vector for this column.
   Let $y^* = 1 - \frac{1}{T} \sum y_t$ and $x^* = \arg\min x_t x_t^t A y_t$.

Claim: $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$.

Proof Idea:
$x_t$ minimizes the best column response is chosen. Clearly good for row.
Column best response is at least what it is against $x_t$.
Total loss, $L$ is at least $\frac{1}{T} \sum y_t$.
Best row payoff, $L^*$ is roughly less than $L$ due to MW analysis.
Combine bounds. Done!
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) $m$ pure row strategies are experts.
   Use multiplicative weights, produce row distribution.
   Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

2) Each day, adversary plays best column response to $x_t$.
   Choose column of $A$ that maximizes row’s expected loss.
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.  

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) $m$ pure row strategies are experts.  
Use multiplicative weights, produce row distribution.  
Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.  

2) Each day, adversary plays best column response to $x_t$.  
Choose column of $A$ that maximizes row’s expected loss.  
Let $y_t$ be indicator vector for this column.

Let $y^*_T = \sum_{t} y_t$ and $x^* = \arg\min x_t x_t^T A y_t$.  

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.  

Proof Idea: $x_t$ minimizes the best column response is chosen.  
Clearly good for row.  
Column best response is at least what it is against $x_t$.  
Total loss, $L$ is at least $\frac{1}{2}$ of column payoff.  
Best row payoff, $L^*$ is roughly less than $L$ due to MW analysis.  
Combine bounds.  
Done!
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) $m$ pure row strategies are experts.  
   Use multiplicative weights, produce row distribution.  
   Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

2) Each day, adversary plays best column response to $x_t$.  
   Choose column of $A$ that maximizes row’s expected loss.  
   Let $y_t$ be indicator vector for this column.

Let $y^* = \frac{1}{T} \sum_t y_t$
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) $m$ pure row strategies are experts.
Use multiplicative weights, produce row distribution.
Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

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Choose column of $A$ that maximizes row’s expected loss.
Let $y_t$ be indicator vector for this column.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$. 
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

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Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$. 
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Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_t A y_t$.

**Claim:** $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$.

**Proof Idea:**
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

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**Claim:** $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$.

Proof Idea:

$x_t$ minimizes the best column response is chosen.
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

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Choose column of $A$ that maximizes row’s expected loss.
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Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Proof Idea:
$x_t$ minimizes the best column response is chosen. Clearly good for row.
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\epsilon^2}$ days:

1) $m$ pure row strategies are experts. 
Use multiplicative weights, produce row distribution. 
Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

2) Each day, adversary plays best column response to $x_t$. 
Choose column of $A$ that maximizes row’s expected loss. 
Let $y_t$ be indicator vector for this column.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$.

Proof Idea:
$x_t$ minimizes the best column response is chosen. Clearly good for row. 
   column best response is at least what it is against $x_t$. 
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

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   Use multiplicative weights, produce row distribution.
   Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

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Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Proof Idea:
$x_t$ minimizes the best column response is chosen. Clearly good for row.
   column best response is at least what it is against $x_t$. Total loss, $L$ is at least column payoff.
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

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Claim: $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$.

Proof Idea:
$x_t$ minimizes the best column response is chosen. Clearly good for row.

Column best response is at least what it is against $x_t$. Total loss, $L$ is at least column payoff. Best row payoff, $L^*$ is roughly less than $L$ due to MW analysis.
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

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Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

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Choose column of $A$ that maximizes row’s expected loss.
Let $y_t$ be indicator vector for this column.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

**Claim:** $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$.

**Proof Idea:**

$x_t$ minimizes the best column response is chosen. Clearly good for row.

Column best response is at least what it is against $x_t$. Total loss, $L$ is at least column payoff. Best row payoff, $L^*$ is roughly less than $L$ due to MW analysis.

Combine bounds.
Games and Experts.

Assume: $A$ has payoffs in $[0, 1]$.

For $T = \frac{\log n}{\epsilon^2}$ days:

1) $m$ pure row strategies are experts.
   Use multiplicative weights, produce row distribution.
   Let $x_t$ be distribution (row strategy) $x_t$ on day $t$.

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**Claim:** $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$.

Proof Idea:

$x_t$ minimizes the best column response is chosen. Clearly good for row.

Column best response is at least what it is against $x_t$. Total loss, $L$ is at least column payoff. Best row payoff, $L^*$ is roughly less than $L$ due to MW analysis.

Combine bounds. Done!
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$. 

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min x_t x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max y x^* A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x_t$.

Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$ - best row against all the columns played.

$\rightarrow$ best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$ → best row against $TA y^*$.

$\rightarrow L^* \leq TR(y^*)$.

Multiplicative Weights: $L \leq (1 + \varepsilon) L^* + \ln n \varepsilon T$ → $C(x^*) \leq (1 + \varepsilon) R(y^*) + \ln n \varepsilon T$.

$T = \ln n \varepsilon^2$, $R(y^*) \leq 1$ → $C(x^*) - R(y^*) \leq 2\varepsilon$. 

Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T}\sum_t y_t$
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$. 
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.
Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y)^*$ are $2\epsilon$-optimal for matrix $A$. 
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$. 
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t Ay_t$.

Claim: $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* Ay$.

Loss on day $t$, $x_t Ay_t \geq C(x^*)$ by the choice of $x$.  

Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_tA y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^*A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq T C(x^*)$. 
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.
Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.
Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

**Claim:** $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.
Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.
Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.
best row against $\sum_t A y_t$
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.

best row against $\sum_t A y_t$ and $Ty^* = \sum_t y_t$
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum y_t$ and $x^* = \arg\min_{x_t} x_tA y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^*Ay$.

Loss on day $t$, $x_tA y_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.

- best row against $\sum y_t$ and $Ty^* = \sum y_t$
- $\rightarrow$ best row against $TAy^*$.
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.

best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$

$\longrightarrow$ best row against $T A y^*$.

$\longrightarrow L^* \leq TR(y^*)$. 
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.

  Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.

  Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.

  best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$
  $\rightarrow$ best row against $T A y^*$.
  $\rightarrow L^* \leq TR(y^*)$. 

Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\epsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.

best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$

$\rightarrow$ best row against $TA y^*$.

$\rightarrow L^* \leq TR(y^*)$.

Multiplicative Weights:
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_t Ay_t$.

**Claim:** $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* Ay$.

Loss on day $t$, $x_t Ay_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$ - best row against all the columns played.

- best row against $\sum_t Ay_t$ and $Ty^* = \sum_t y_t$
  $\rightarrow$ best row against $TAy^*$.
  $\rightarrow L^* \leq TR(y^*)$.

Multiplicative Weights: $L \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon}$
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.

best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$

$\rightarrow$ best row against $T A y^*$.

$\rightarrow L^* \leq TR(y^*)$.

Multiplicative Weights: $L \leq (1 + \varepsilon) L^* + \frac{\ln n}{\varepsilon}$

$TC(x^*) \leq (1 + \varepsilon) TR(y^*) + \frac{\ln n}{\varepsilon}$
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.

Loss on day $t$, $x_t A y_t \geq C(x^*)$ by the choice of $x$.

Thus, algorithm loss, $L$, is $\geq T C(x^*)$.

Best expert: $L^*$- best row against all the columns played.

best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$

$\to$ best row against $T A y^*$.

$\to$ $L^* \leq T R(y^*)$.

Multiplicative Weights: $L \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon}$

$T C(x^*) \leq (1 + \varepsilon) T R(y^*) + \frac{\ln n}{\varepsilon} \to C(x^*) \leq (1 + \varepsilon) R(y^*) + \frac{\ln n}{\varepsilon T}$
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t Ay_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* Ay$.
Loss on day $t$, $x_t Ay_t \geq C(x^*)$ by the choice of $x$.
Thus, algorithm loss, $L$, is $\geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.

  best row against $\sum_t Ay_t$ and $Ty^* = \sum_t y_t$
  → best row against $TAy^*$.
  → $L^* \leq TR(y^*)$.

Multiplicative Weights: $L \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon}$

$TC(x^*) \leq (1 + \varepsilon)TR(y^*) + \frac{\ln n}{\varepsilon} \rightarrow C(x^*) \leq (1 + \varepsilon)R(y^*) + \frac{\ln n}{\varepsilon T}$

$\rightarrow C(x^*) - R(y^*) \leq \varepsilon R(y^*) + \frac{\ln n}{\varepsilon T}$. 
Approximate Equilibrium!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $y^* = \frac{1}{T} \sum_t y_t$ and $x^* = \arg\min_{x_t} x_t A y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

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Best expert: $L^*$ - best row against all the columns played.

best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$

$\rightarrow$ best row against $T A y^*$.

$\rightarrow$ $L^* \leq TR(y^*)$.

Multiplicative Weights: $L \leq (1 + \varepsilon) L^* + \frac{\ln n}{\varepsilon}$

$TC(x^*) \leq (1 + \varepsilon) TR(y^*) + \frac{\ln n}{\varepsilon} \rightarrow C(x^*) \leq (1 + \varepsilon) R(y^*) + \frac{\ln n}{\varepsilon T}$

$\rightarrow C(x^*) - R(y^*) \leq \varepsilon R(y^*) + \frac{\ln n}{\varepsilon T}$.

$T = \frac{\ln n}{\varepsilon^2}$, $R(y^*) \leq 1$
Approximate Equilibrium!

Experts: \( x_t \) is strategy on day \( t \), \( y_t \) is best column against \( x_t \).

Let \( y^* = \frac{1}{T} \sum_t y_t \) and \( x^* = \arg\min_{x_t} x_tAy_t \).

Claim: \((x^*, y)^*\) are \(2\varepsilon\)-optimal for matrix \( A \).

Column payoff: \( C(x^*) = \max_y x^*Ay \).
Loss on day \( t \), \( x_tAy_t \geq C(x^*) \) by the choice of \( x \).
Thus, algorithm loss, \( L \), is \( \geq TC(x^*) \).

Best expert: \( L^* \)- best row against all the columns played.

best row against \( \sum_t Ay_t \) and \( Ty^* = \sum_t y_t \)
\( \rightarrow \) best row against \( TAy^* \).
\( \rightarrow L^* \leq TR(y^*) \).

Multiplicative Weights: \( L \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon} \)

\( TC(x^*) \leq (1 + \varepsilon)TR(y^*) + \frac{\ln n}{\varepsilon} \rightarrow C(x^*) \leq (1 + \varepsilon)R(y^*) + \frac{\ln n}{\varepsilon T} \)
\( \rightarrow C(x^*) - R(y^*) \leq \varepsilon R(y^*) + \frac{\ln n}{\varepsilon T} \).

\( T = \frac{\ln n}{\varepsilon^2} \), \( R(y^*) \leq 1 \)
\( \rightarrow C(x^*) - R(y^*) \leq 2\varepsilon \).
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$. 

Let $x^* = \frac{1}{T} \sum_t x_t$ and $y^* = \frac{1}{T} \sum_t y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_{y} x^* A y$.

Let $y_r$ be best response to $C(x^*)$.

Day $t$, $y_t$ best response to $x_t \rightarrow x_t A y_t \geq x_t A y_r$.

Algorithm loss: $\sum_t x_t A y_t \geq \sum_t x_t A y_r \implies L \geq TC(x^*)$.

Best expert: $L^*$ - best row against all the columns played.

best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t \rightarrow$ best row against $TA y^*$.

$L^* \leq TR(y^*)$.

Multiplicative Weights: $L \leq (1 + \epsilon)L^* + \ln n \epsilon T C(x^*) \leq (1 + \epsilon)R(y^*) + \ln n \epsilon T \rightarrow C(x^*) \leq (1 + \epsilon)R(y^*) + \ln n \epsilon T \leq 2\varepsilon$. 

$T = \ln n \epsilon^2$, $R(y^*) \leq 1 \rightarrow C(x^*) - R(y^*) \leq 2\varepsilon$. 

Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $x^* = \frac{1}{T} \sum_t x_t$
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $x^* = \frac{1}{T} \sum_t x_t$ and $y^* = \frac{1}{T} \sum_t y_t$. 
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

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Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$. 
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $x^* = \frac{1}{T} \sum_t x_t$ and $y^* = \frac{1}{T} \sum_t y_t$.

Claim: $(x^*, y^*)$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* A y$.
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $x^* = \frac{1}{T} \sum_t x_t$ and $y^* = \frac{1}{T} \sum_t y_t$.

Claim: $(x^*, y)^*$ are $2\varepsilon$-optimal for matrix $A$.

Column payoff: $C(x^*) = \max_y x^* Ay$.

Let $y_r$ be best response to $C(x^*)$. 

Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

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Column payoff: $C(x^*) = \max_y x^* A y$.
   Let $y_r$ be best response to $C(x^*)$.
   Day $t, y_t$ best response to $x_t \rightarrow x_t A y_t \geq x_t A y_r$.
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

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Column payoff: $C(x^*) = \max_y x^*Ay$.

Let $y_r$ be best response to $C(x^*)$.

Day $t$, $y_t$ best response to $x_t \rightarrow x_tAy_t \geq x_tAy_r$.

Algorithm loss: $\sum_t x_tAy_t \geq \sum_t x_tAy_r$.
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

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Let $y_r$ be best response to $C(x^*)$.

Day $t$. $y_t$ best response to $x_t \rightarrow x_t A y_t \geq x_t A y_r$.

Algorithm loss: $\sum_t x_t A y_t \geq \sum_t x_t A y_r$

$L \geq TC(x^*)$. 

Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $x^* = \frac{1}{T} \sum_t x_t$ and $y^* = \frac{1}{T} \sum_t y_t$.

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    Algorithm loss: $\sum_t x_t Ay_t \geq \sum_t x_t Ay_r$
    $L \geq TC(x^*)$.

Best expert: $L^*$- best row against all the columns played.
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $x^* = \frac{1}{T} \sum_t x_t$ and $y^* = \frac{1}{T} \sum_t y_t$.

**Claim:** $(x^*, y)^*$ are $2\epsilon$-optimal for matrix $A$.

**Column payoff:** $C(x^*) = \max_y x^* Ay$.
- Let $y_r$ be best response to $C(x^*)$.
- Day $t$, $y_t$ best response to $x_t \rightarrow x_t Ay_t \geq x_t Ay_r$.
- Algorithm loss: $\sum_t x_t Ay_t \geq \sum_t x_t Ay_r$
  $$L \geq TC(x^*).$$

Best expert: $L^*$- best row against all the columns played.
- best row against $\sum_t Ay_t$
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

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Approximate Equilibrium: notes!

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best row against $\sum_t Ay_t$ and $Ty^* = \sum_t y_t$

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$\rightarrow L^* \leq TR(y^*)$. 

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Best expert: $L^*$ - best row against all the columns played.

- best row against $\sum_t A y_t$ and $T y^* = \sum_t y_t$
- $\rightarrow$ best row against $T A y^*$.
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Multiplicative Weights:
Approximate Equilibrium: notes!

Experts: \(x_t\) is strategy on day \(t\), \(y_t\) is best column against \(x_t\).

Let \(x^* = \frac{1}{T} \sum_t x_t\) and \(y^* = \frac{1}{T} \sum_t y_t\).

Claim: \((x^*, y^*)\) are \(2\varepsilon\)-optimal for matrix \(A\).

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Day \(t\), \(y_t\) best response to \(x_t \rightarrow x_t Ay_t \geq x_t Ay_r\).

Algorithm loss: \(\sum_t x_t Ay_t \geq \sum_t x_t Ay_r\)

\(L \geq TC(x^*)\).

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Multiplicative Weights: \(L \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon}\)
Approximate Equilibrium: notes!

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$TC(x^*) \leq (1 + \varepsilon)TR(y^*) + \frac{\ln n}{\varepsilon}$
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\[ L \geq TC(x^*) \]

Best expert: \( L^* \)- best row against all the columns played.

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Multiplicative Weights: \( L \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon} \)

\[ TC(x^*) \leq (1 + \varepsilon)TR(y^*) + \frac{\ln n}{\varepsilon} \rightarrow C(x^*) \leq (1 + \varepsilon)R(y^*) + \frac{\ln n}{\varepsilon T} \]
Approximate Equilibrium: notes!

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Algorithm loss: $\sum_t x_t Ay_t \geq \sum_t x_t Ay_r$

$L \geq TC(x^*)$.

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Best expert: \( L^* \)- best row against all the columns played.

best row against \( \sum_t A y_t \) and \( Ty^* = \sum_t y_t \)
\[ \rightarrow \text{best row against } TAy^* \]
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Multiplicative Weights: \( L \leq (1 + \varepsilon)L^* + \frac{\ln n}{\varepsilon} \)

\[ TC(x^*) \leq (1 + \varepsilon)TR(y^*) + \frac{\ln n}{\varepsilon} \rightarrow C(x^*) \leq (1 + \varepsilon)R(y^*) + \frac{\ln n}{\varepsilon T} \]
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\( T = \frac{\ln n}{\varepsilon^2}, R(y^*) \leq 1 \)
Approximate Equilibrium: notes!

Experts: $x_t$ is strategy on day $t$, $y_t$ is best column against $x_t$.

Let $x^* = \frac{1}{T} \sum_t x_t$ and $y^* = \frac{1}{T} \sum_t y_t$.

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$TC(x^*) \leq (1 + \epsilon)TR(y^*) + \frac{\ln n}{\epsilon} \rightarrow C(x^*) \leq (1 + \epsilon)R(y^*) + \frac{\ln n}{\epsilon T}$

$\rightarrow C(x^*) - R(y^*) \leq \epsilon R(y^*) + \frac{\ln n}{\epsilon T}$.

$T = \frac{\ln n}{\epsilon^2}$, $R(y^*) \leq 1 \rightarrow C(x^*) - R(y^*) \leq 2\epsilon$. 
For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.
Comments

For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium. Does an equilibrium exist?
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Does an equilibrium exist? Yes.
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Something about math here?
For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.

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Something about math here? Fixed point theorem.
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Later: will use geometry, linear programming.
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Complexity?
For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.

Does an equilibrium exist? Yes.

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Complexity?

$$T = \frac{\ln n}{\varepsilon^2}$$
Comments

For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.  
Does an equilibrium exist? Yes. 
Something about math here? Fixed point theorem. 
Later: will use geometry, linear programming. 
Complexity? 
$$T = \frac{\ln n}{\varepsilon^2} \rightarrow O(nm\frac{\log n}{\varepsilon^2}).$$
For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.

Does an equilibrium exist? Yes.

Something about math here? Fixed point theorem.

Later: will use geometry, linear programming.

Complexity?

$$T = \frac{\ln n}{\varepsilon^2} \to O(nm^{\log n/\varepsilon^2}).$$

Basically linear!
For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.

Does an equilibrium exist? Yes.

Something about math here? Fixed point theorem.

Later: will use geometry, linear programming.

Complexity?

$$T = \frac{\ln n}{\varepsilon^2} \to O(nm\frac{\log n}{\varepsilon^2}).$$

Basically linear!

Versus Linear Programming: $O(n^3 m)$
For any $\epsilon$, there exists an $\epsilon$-Approximate Equilibrium.

Does an equilibrium exist? Yes.

Something about math here? Fixed point theorem.

Later: will use geometry, linear programming.

Complexity?

$$T = \frac{\ln n}{\epsilon^2} \rightarrow O(nm\frac{\log n}{\epsilon^2}).$$ Basically linear!

Versus Linear Programming: $O(n^3 m)$ Basically quadratic.
For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.

Does an equilibrium exist? Yes.

Something about math here? Fixed point theorem.

Later: will use geometry, linear programming.

Complexity?

$$T = \frac{\ln n}{\varepsilon^2} \to O(nm\frac{\log n}{\varepsilon^2}).$$

Basically linear!

Versus Linear Programming: $O(n^3m)$ Basically quadratic.

(Faster linear programming: $O(\sqrt{n + m})$ linear solution solves.)
For any $\varepsilon$, there exists an $\varepsilon$-Approximate Equilibrium.

Does an equilibrium exist? Yes.

Something about math here? Fixed point theorem.

Later: will use geometry, linear programming.

Complexity?

$$T = \frac{\ln n}{\varepsilon^2} \to O(nm\frac{\log n}{\varepsilon^2}).$$ Basically linear!

Versus Linear Programming: $O(n^3m)$ Basically quadratic.

(Faster linear programming: $O(\sqrt{n+m})$ linear solution solves.)

Still much slower
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Does an equilibrium exist? Yes.

Something about math here? Fixed point theorem.

Later: will use geometry, linear programming.

Complexity?

$$T = \frac{\ln n}{\varepsilon^2} \rightarrow O(nm\frac{\log n}{\varepsilon^2}).$$  Basically linear!

Versus Linear Programming: $O(n^3m)$ Basically quadratic.

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Remarks

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“In practice.”
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.
Toll/Congestion

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Matrix:
row for each routing: \( r \)
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**Offense: (Best Response.)**
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**Offense:** *(Best Response.)*
Router: route along shortest paths.
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**Offense: (Best Response.)** 
Router: route along shortest paths.  
Toll: charge most loaded edge.
Toll/Congestion

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**Offense:** *(Best Response.)*
**Router:** route along shortest paths.
**Toll:** charge most loaded edge.

**Defense:** **Toll:** maximize shortest path under tolls.
Toll/Congestion

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**Offense: (Best Response.)**
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**Toll:** charge most loaded edge.

**Defense: **Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.
Toll/Congestion

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**Offense: (Best Response.)**
Router: route along shortest paths.
Toll: charge most loaded edge.

**Defense:** Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.
Two person game.

Row is router.

Version with row and column flipped may work.
Two person game.

Row is router.

An exponential number of rows.
Two person game.

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Two person game with experts won’t be so easy to implement.
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Next Time.