Linear Systems...

Linear Systems

$Ax = b$

Find x.

Gaussian elimination: $O(n^3)$
$O(n^{2.36...})$ with fast matrix multiplication.

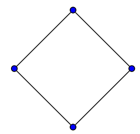
Iterative Methods: $O(nm \log \frac{1}{\varepsilon})$ to ε approximate.
For today: where m is sum of nonzeros in matrix.
For positive semidefinite matrix.

Today: $\tilde{O}(m)$ for Laplacian matrices.
Laplacian: $dI - A$ where A is adjacency matrix of a graph.
$\rightarrow$ symmetric diagonally dominant matrices by reduction.

Electrical Flow: a detour.

A graph $G = (V,E)$.
Circuit: nodes V, resistors E, value 1 (for today.)

Given $\chi : V \rightarrow \Re$
Find flow that routes χ and minimizes
$\sum_e f(e)^2$.



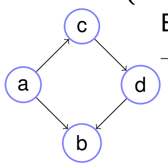
Claim: Minimizer is electrical flow.

Flow corresponds to flow induced by a set of potentials.

Some Matrices.

Given $G = (V,E)$, arbitrarily orient edges.

$$B_{v,e} = \begin{cases} -1 & e = (u,v) \\ 1 & e = (v,u) \\ 0 & \text{otherwise} \end{cases} \qquad L_{u,v} = \begin{cases} d & u = v \\ -1 & (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$



B

	a	b	c	d
(a,b)	1	-1	0	0
(a,c)	1	0	-1	0
(c,d)	0	0	1	-1
(d,b)	0	-1	0	1

L

	a	b	c	d
a	2	-1	-1	0
b	-1	2	0	-1
c	-1	0	2	-1
d	0	-1	-1	2

Fun facts: $\mathbf{f} \in \Re^{|E|}$
$[B^T f]_u = \sum_{e=(u,v)} f_e - \sum_{e=(v,u)} f_e$
$B^T B = L$
$[Bx]_{e=(u,v)} = x_u - x_v$
$x^T L x = \sum_{e=(u,v)} (x_u - x_v)^2$

Duality..

Given G, χ, $\chi \perp 1$

Minimize $|f|^2$ subject to $B^T f = \chi$.

Lagrangian: $L(\phi, f) = \sum_e f(e)^2 + 2\phi^T(\chi - B^T f)$

Lagrangian Dual: Find ϕ that maximizes $\min_f L(\phi, f)$.

Given ϕ, minimize $L(\phi, f)$? Calculus.

For $e = (u,v)$
$2f(e) + 2(\phi_v - \phi_u) = 0$ (Minimum when partial derivatives = 0.)

$\rightarrow f(e) = (\phi_u - \phi_v)$ Potential differences!!!

Matrix Form: $f = B\phi$ Again, flows should be potential differences.

Dual problem: Find ϕ that maximizes ...
$\max_\phi 2\phi^T \chi - \phi^T L \phi$
Note: want $\phi^T L \phi = \sum_e (\phi_u - \phi_v)^2$ to be small.
Minimize Squared Potential differences!

Why did we take dual?

Dual problem:
Find ϕ that maximizes ...
$\max_\phi 2\phi\chi - \phi L \phi$

Take the derivative:
$L\phi - \chi$
$L\phi = \chi$ at optimal point!

Optimal potential is solution to a Laplacian linear system.

Also useful for convergence.

Algorithm maintains feasible ϕ, f,

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi^T L \phi$

Duality gap is "distance" from optimal!

Algorithm: Work on flow and potentials.
To drive gap to 0.

Alg.

Given: χ, G

Take a spanning tree T of G. (Which tree?)

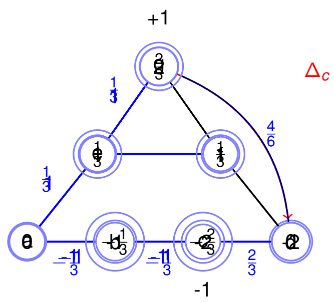
Route flow, f, to satisfy χ through T
Compute, ϕ, using tree ; $\phi_s = 0$, add f_e through T

Repeat:
Choose non-tree edge $e = (u,v)$ (Which non-tree edge?)
$f(e) = (\phi_u - \phi_v)/(l_T(u,v)+1)$
($l_T(u,v)$ path length in T)
Route excess on path through tree.

Which Tree?
Claim: Linear time algorithm for T w/ stretch $O(m \log n \log\log n)$!
Stretch: $\sum_{e=(u,v)} l_T(u,v)$

Which non-tree edge?
Choose an edge w/prob. proportional to $l_T(e)$.

Finds $(1+\varepsilon)$ approximation in $O(m \log n \log\log n \log(\frac{n}{\varepsilon}))$!



Energy reduction.

Given T,$e = (u,v)$, let $R_e = l_T(u,v)+1$.

Algorithm:

Repeatedly "Fix" edge $e = (u,v)$.
Route $-\delta = -\frac{\sum_{e' \in C_e} f(e')}{R_e}$ flow around cycle induced in T: C_e
(assume e' are oriented around cycle.)

Difference in energy from f and f'.
$\sum_{e' \in C_e} (f(e') - \delta)^2 - (f(e'))^2 = \sum_{e' \in C_e} -2f(e')\delta + \delta^2$
$= -(2\delta \sum_{e' \in C_e} f(e')) + R_E \delta^2$

Note: $\sum_{e' \in C_e} f(e') = R_e \delta$

$\rightarrow -\Delta^2_{C_e}/R_e$ where $\Delta_{C_e} = \sum_{e' \in C_e} f_(e')$.

Fix a part of the potential difference, Δ_{c_e} around cyle!!

$\rightarrow$ reduction of $\Delta^2_{C_e}/R_e$ in energy!

Fix $1/R_e$ of a cycle violation!

Duality Gap?

Algorithm maintains feasible ϕ, f, $(B^T f = \chi)$

Primal value: $|f|^2$.
Dual value: $2\phi\chi - \phi L \phi$

ϕ is tree induced voltages.

Total Duality Gap?

Gap: $|f|^2 - (2\phi^T \chi - \phi^T L \phi)$.
$= |f|^2 - 2\phi^T B^T f + \phi^T B^T B \phi$ where $B^T f = \chi$ and $L = B^T B$.
$= (f - B\phi)^T (f - B\phi)$.

Gap $= \sum_e (f(e) - \Delta_\phi(e))^2$ Difference between ϕ flow and f.

$\Delta_\phi(u,v) = \sum_{e \in P_{u,v}} -f(e)$. assume $f(e)$ is oriented around cycle.

For $e \in T$, $\Delta_\phi(e) = 0$. For $e \notin T$ $f(e) + \sum_{e \in P_e} f(e) = \Delta_{c_e}(f)$

Duality Gap: $\sum_{e \notin T} \sum_e \Delta_{c_e}(f)^2$

Total distance from optimal is cycle violations!

Claim: E[change in energy$|Gap$] $= \frac{Gap}{\tau}$ (τ is stretch of E in T.)

Duality Gap: $\sum_{e \notin T} \Delta_{C_e}(f)^2$

Choose edge e reduce energy by $-\frac{\Delta^2_{C_e}}{R_e}$.

Choose edge with probability $\frac{R_e}{\tau}$.

Expected reduction $-\sum_e \frac{R_e}{\tau} \frac{\Delta^2_{C_e}}{R_e} = -\sum_e \frac{\Delta^2_{C_e}}{\tau}$ □

Duality Gap reduces by $(1 - 1/\tau)$ every iteration on expectation.

$O(\tau \log(n/\varepsilon))$ iterations gives $(1+\varepsilon)$ approximation.

$\tau = O(m \log n \log\log n)$ ...

$\tilde{O}(m)$ iterations

Iteration in $O(\log^2 n)$ time using balanced binary trees.

$\rightarrow \tilde{O}(m)$ time! ! ! ! ! ! ! ! ! ! ! !

See you ...

Thursday