Welcome back...

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Approximate metric on trees?

Approximate metric using a tree.

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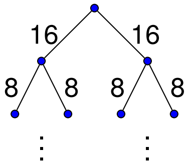

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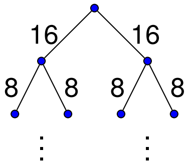

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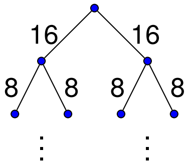

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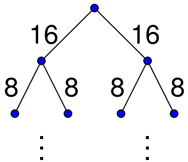

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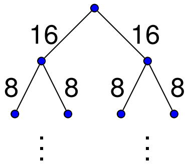

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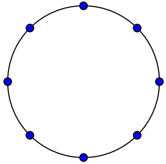

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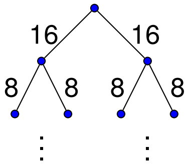

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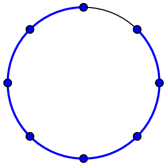

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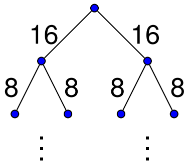

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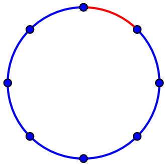

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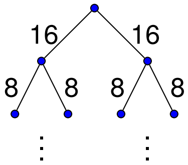

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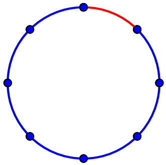

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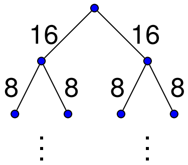

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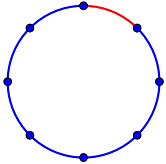

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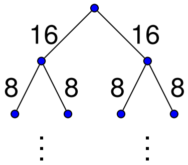

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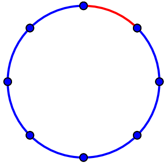

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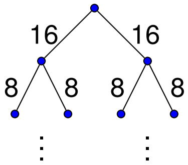

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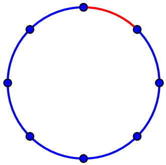

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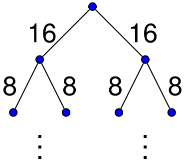

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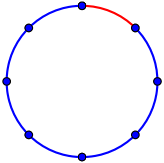

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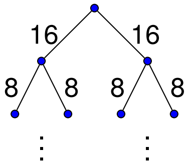

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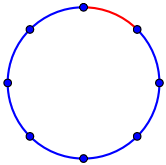

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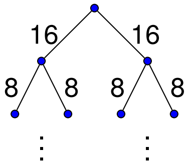

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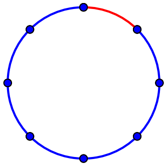

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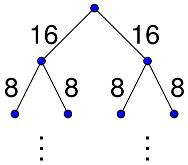

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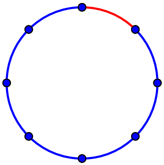

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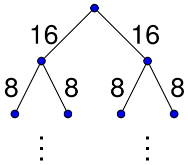

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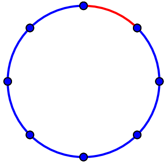

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General metrics?

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Algorithm

Algorithm: (X, d), $\text{diam}(X) \leq D$, $|X| = n$, $d(i, j) \geq 1$

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1. π – random permutation of X.

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occurs with prob. $\frac{d(x,y)}{\Delta/4} = \frac{4d(x,y)}{\Delta}$.

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$d_T(x, y)$ if cut level Δ is 2Δ.

Analysis: (x,y)

Have $Pr[x,y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x,y)}{\beta\Delta} \leq 4\Delta$
(Only consider cut by x, factor 2 loss.)

At level Δ

At some point x is in some Δ level ball.
Renumber nodes in order of distance from x.

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The expected stretch of any pair is $O(\log n)$.

Alternative to Cheeger for expansion.

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If not...a bit more work...

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$\rightarrow O(\log n)$ approximation.

See you ...

Tuesday.