Welcome back...
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Metric spaces.

A metric space $X$, $d(i,j)$ where $d(i,j) \leq d(i,k) + d(k,l)$ and $d(i,j) = d(j,i)$
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Which are metric spaces?

(A) $X$ from $R^d$ and $d(\cdot, \cdot)$ is Euclidean distance.

(B) $X$ from $R^d$ and $d(\cdot, \cdot)$ is squared Euclidean distance.

(C) $X$- vertices in graph, $d(i,j)$ is shortest path distances in graph.

(D) $X$ is a set of vectors and $d(u,v)$ is $u \cdot v$. 
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Input to TSP, facility location, some layout problems, ..., metric labelling.
Hard problems.
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Approximate metric on trees?
Approximate metric using a tree.

Tree metric:

- No distance shrinks. (dominating)
- Every distance stretches \( \leq \alpha \) in expectation.

Map metric onto tree?

Distance 1 goes to \( n - 1! \). Bummer.

Fix it up chappie!

For cycle, remove a random edge get a tree.

Stretch of edge: \( n - 1 \times n + 1 \) \( n \times (n - 1) \approx 2 \).
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Tree metric:
- $X$ is nodes of tree with edge weights $d_T(i,j)$ shortest path metric on tree.
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Probabilistic Tree embedding.

Map $X$ into tree.

Today: the tree will be Hierarchically well-separated (HST).

On Tuesday: use spanning tree for graphical metrics.

The Idea: 

HST ≡ recursive decomposition of metric space.

Decompose space by diameter $\approx \Delta$ balls.

Recurse on each ball for $\Delta / 2$.

Use randomness in selection of ball centers.

the $\approx$ diameter of the balls.
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Algorithm

Algorithm: \((X, d)\), diam\((X)\) ≤ \(D\), \(|X| = n\), \(d(i, j) \geq 1\)
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```python
def subtree(S, \Delta):
    T = []
    if \Delta < 1 return [S]
    foreach i in \(\pi\):
        if i \(\in\) S
            \(B = \text{ball}(i, \beta \Delta)\);
            S = S \setminus B
        T.append(B)
    return map (\lambda x: subtree(x, \Delta / 2), T);
```

3. \(\text{subtree}(X, \Delta)\)

Tree has internal node for each level of call.
Tree edges have weight \(\Delta / 2\) to children.

Claim 1: \(d_T(x, y) \geq d(x, y)\).
\(d(x, y)\) are in different sets at level \(\Delta \leq d(x, y)\).

\(d(x, y) \geq \Delta\)
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   foreach \(i\) in \(\pi\):
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       \(B = \text{ball}(i, \beta \Delta)\); \(S = S / B\)
       \(T\).append\(B\)
   return map (\(\lambda x: \text{subtree}(x, \Delta / 2), T\));
3. subtree\((X, \Delta)\)

Tree has internal node for each level of call. Tree edges have weight \(\Delta / 2\) to children.
Algorithm

Algorithm: \((X, d), \text{diam}(X) \leq D, |X| = n, d(i, j) \geq 1\)

1. \(\pi\) – random permutation of \(X\).
2. Choose \(\beta\) in \([\frac{1}{4}, \frac{1}{2}]\).
   
   ```python
   def subtree(S, \Delta):
       T = []
       if \(\Delta < 1\) return [S]
       foreach i in \(\pi\):
           if \(i \in S\):
               \(B = \text{ball}(i, \beta \Delta)\); \(S = S / B\)
               T.append(B)
       return map (\(\lambda x: \text{subtree}(x, \Delta / 2)\), T);
   ```
3. \(\text{subtree}(X, \Delta)\)

Tree has internal node for each level of call. Tree edges have weight \(\Delta / 2\) to children.

Claim 1: \(d_T(x, y) \geq d(x, y)\).
Algorithm

Algorithm: $(X, d), \text{diam}(X) \leq D$, $|X| = n$, $d(i, j) \geq 1$

1. $\pi$ – random permutation of $X$.
2. Choose $\beta$ in $[\frac{1}{4}, \frac{1}{2}]$.
   
   def subtree(S, $\Delta$):
   
   T = []
   if $\Delta < 1$ return [S]
   foreach i in $\pi$:
   
   if $i \in S$:
   
   \hspace{1em} $B = \text{ball}(i, \beta \Delta) \ ; \ S = S \setminus B$
   \hspace{1em} T.append($B$)
   
   return map ($\lambda$ x: subtree(x, $\Delta/2$), T);

3. subtree($X, \Delta$)

Tree has internal node for each level of call. Tree edges have weight $\Delta/2$ to children.

Claim 1: $d_T(x, y) \geq d(x, y)$.

$d(x, y)$ are in different sets at level $\Delta \leq d(x, y)$. 
Algorithm

Algorithm: \((X, d), \text{diam}(X) \leq D, |X| = n, d(i, j) \geq 1\)

1. \(\pi\) – random permutation of \(X\).
2. Choose \(\beta\) in \([\frac{1}{4}, \frac{1}{2}]\).
   
   \[
   \text{def subtree}(S, \Delta) :
   \]
   
   \[
   T = []
   \]
   
   if \(\Delta < 1\) return [S]
   
   foreach \(i\) in \(\pi\):
   
   if \(i \in S\)
   
   \[
   B = \text{ball}(i, \beta \Delta) ; S = S / B
   \]
   
   T.append(B)
   
   return map (\(\lambda\) \(x\): subtree(\(x, \Delta/2\), T));
3. subtree(\(X, \Delta\))

Tree has internal node for each level of call. Tree edges have weight \(\Delta/2\) to children.

**Claim 1:** \(d_T(x, y) \geq d(x, y)\).

\(d(x, y)\) are in different sets at level \(\Delta \leq d(x, y)\).

\[\rightarrow d(x, y) \geq \Delta\]
Analysis: idea

**Claim:** \( E[d_T(x, y)] = O(\log n) d(x, y) \).
Analysis: idea

**Claim:** \( E[d_T(x, y)] = O(\log n) d(x, y) \).

Cut at level \( \Delta \rightarrow d_T(x, y) \approx 2\Delta \).
Analysis: idea

Claim: \( E[d_T(x, y)] = O(\log n)d(x, y) \).
Cut at level \( \Delta \rightarrow d_T(x, y) \approx 2\Delta \). (Level of subtree call.)

\[ \Pr\left[ \text{cut at level } \Delta \right] ? \]
Would like it to be \( d(x, y) \Delta \).
→ expected length is \( \sum \Delta = D/2 \) \( d(x, y) \Delta = 2d(x, y) \Delta \).
Why should it be \( d(x, y) \Delta \)?
smaller the edge the less likely to be on edge of ball.
larger the \( \Delta \), more room inside ball.
random diameter jiggles edge of ball.
→ \( \Pr\left[ x, y \text{ cut by ball } | x \text{ in ball} \right] \approx d(x, y) \beta \Delta \leq 4 \Delta \).

The problem?
Could be cut be many different balls.
For each probability is good, but could be hit by many.
random permutation to deal with this
Analysis: idea

Claim: $E[d_T(x, y)] = O(\log n)d(x, y)$.

Cut at level $\Delta \rightarrow d_T(x, y) \approx 2\Delta$. (Level of subtree call.)

$Pr[\text{cut at level} \Delta]$?

Why should it be $d(x, y)\Delta$?

smaller the edge the less likely to be on edge of ball.
larger the $\Delta$, more room inside ball.
random diameter jiggles edge of ball.

$Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx d(x, y)\beta \Delta \leq 4\Delta$

The problem?

Could be cut be many different balls.
For each probability is good, but could be hit by many.
random permutation to deal with this
**Analysis: idea**

**Claim:** $E[d_T(x, y)] = O(\log n)d(x, y)$.

Cut at level $\Delta \rightarrow d_T(x, y) \approx 2\Delta$. (Level of subtree call.)

$Pr[\text{cut at level } \Delta]$?

- Would like it to be $\frac{d(x, y)}{\Delta}$.
Analysis: idea

**Claim:** $E[d_T(x, y)] = O(\log n)d(x, y)$.

Cut at level $\Delta \rightarrow d_T(x, y) \approx 2\Delta$. (Level of subtree call.)

$Pr[\text{cut at level}\Delta]$?

Would like it to be $\frac{d(x, y)}{\Delta}$.

$\rightarrow$ expected length is $\sum_{\Delta=D/2}^{D} 2\Delta \frac{d(x, y)}{\Delta} = 2d(x, y)$. 

Why should it be $\frac{d(x, y)}{\Delta}$?

smaller the edge the less likely to be on edge of ball.
larger the $\delta$, more room inside ball.

$\rightarrow$ $Pr[\text{x, y cut by ball | x in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta$.

The problem?

Could be cut be many different balls.

For each probability is good, but could be hit by many.

random permutation to deal with this
Analysis: idea

**Claim:** $E[d_T(x, y)] = O(\log n)d(x, y)$.

Cut at level $\Delta \rightarrow d_T(x, y) \approx 2\Delta$. (Level of subtree call.)

$Pr[\text{cut at level } \Delta]$?

Would like it to be $\frac{d(x, y)}{\Delta}$.

$\rightarrow$ expected length is $\sum_{\Delta=\Delta/2^i} (2\Delta) \frac{d(x, y)}{\Delta} = 2d(x, y)$.

Why should it be $\frac{d(x, y)}{\Delta}$?
Analysis: idea

**Claim:** \( E[d_T(x, y)] = O(\log n) d(x, y) \).

Cut at level \( \Delta \rightarrow d_T(x, y) \approx 2\Delta \). (Level of subtree call.)

\[ Pr[\text{cut at level} \Delta] = \frac{d(x, y)}{\Delta} \]

\[ \rightarrow \text{expected length is } \sum_{\Delta=D/2}^{\Delta} (2\Delta)\frac{d(x, y)}{\Delta} = 2d(x, y). \]

Why should it be \( \frac{d(x, y)}{\Delta} \)?

- smaller the edge the less likely to be on edge of ball.
- larger the \( \Delta \), more room inside ball.
- random diameter jiggles edge of ball.

\[ \Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x, y)}{\Delta} \triangleq 4\Delta \]

The problem? Could be cut be many different balls.

For each probability is good, but could be hit by many.

random permutation to deal with this
Analysis: idea

Claim: \( E[d_T(x, y)] = O(\log n) d(x, y). \)

Cut at level \( \Delta \) \( \rightarrow \) \( d_T(x, y) \approx 2\Delta. \) (Level of subtree call.)

\( Pr[\text{cut at level} \Delta]? \)

Would like it to be \( \frac{d(x, y)}{\Delta}. \)

\( \rightarrow \) expected length is \( \sum_{\Delta=D/2}^{D} (2\Delta) \frac{d(x, y)}{\Delta} = 2d(x, y). \)

Why should it be \( \frac{d(x, y)}{\Delta}? \)

smaller the edge the less likely to be on edge of ball.
larger the delta, more room inside ball.
Analysis: idea

**Claim:** $E[d_T(x, y)] = O(\log n)d(x, y)$.

Cut at level $\Delta \rightarrow d_T(x, y) \approx 2\Delta$. (Level of subtree call.)

$Pr[\text{cut at level } \Delta]$?

Would like it to be $\frac{d(x, y)}{\Delta}$.

$\rightarrow$ expected length is $\sum_{\Delta=D/2^i}(2\Delta)\frac{d(x, y)}{\Delta} = 2d(x, y)$.

Why should it be $\frac{d(x, y)}{\Delta}$?

smaller the edge the less likely to be on edge of ball.
larger the delta, more room inside ball.

random diameter jiggles edge of ball.
**Analysis: idea**

**Claim:** $E[d_T(x, y)] = O(\log n)d(x, y)$.

Cut at level $\Delta \rightarrow d_T(x, y) \approx 2\Delta$. (Level of subtree call.)

$Pr[\text{cut at level } \Delta]$?

Would like it to be $\frac{d(x,y)}{\Delta}$.

$\rightarrow$ expected length is $\sum_{\Delta = D/2^i}(2\Delta)\frac{d(x,y)}{\Delta} = 2d(x, y)$.

Why should it be $\frac{d(x,y)}{\Delta}$?

smaller the edge the less likely to be on edge of ball.
larger the delta, more room inside ball.
  random diameter jiggles edge of ball.

$\rightarrow Pr[x, y \text{ cut by ball}| x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta$
Analysis: idea

Claim: \( E[d_T(x, y)] = O(\log n) d(x, y) \).
Cut at level \( \Delta \rightarrow d_T(x, y) \approx 2\Delta \). (Level of subtree call.)
\( Pr[\text{cut at level } \Delta] \)?

Would like it to be \( \frac{d(x,y)}{\Delta} \).

\( \rightarrow \) expected length is \( \sum_{\Delta=D/2} (2\Delta) \frac{d(x,y)}{\Delta} = 2d(x, y) \).

Why should it be \( \frac{d(x,y)}{\Delta} \)?
smaller the edge the less likely to be on edge of ball.
larger the delta, more room inside ball.
   random diameter jiggles edge of ball.

\( \rightarrow Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta \)
The problem?
Analysis: idea

**Claim:** $E[d_T(x, y)] = O(\log n)d(x, y)$.

Cut at level $\Delta \rightarrow d_T(x, y) \approx 2\Delta$. (Level of subtree call.)

$Pr[\text{cut at level } \Delta]$?

Would like it to be $\frac{d(x, y)}{\Delta}$.

$\rightarrow$ expected length is $\sum_{\Delta=\Delta/2^i}(2\Delta)\frac{d(x, y)}{\Delta} = 2d(x, y)$.

Why should it be $\frac{d(x, y)}{\Delta}$?

smaller the edge the less likely to be on edge of ball.
larger the delta, more room inside ball.
random diameter jiggles edge of ball.

$\rightarrow Pr[x, y \text{ cut by ball}\mid x \text{ in ball}] \approx \frac{d(x, y)}{\beta\Delta} \leq 4\Delta$

The problem?
Could be cut be many different balls.
Analysis: idea

**Claim:** \( E[d_T(x, y)] = O(\log n) d(x, y) \).

Cut at level \( \Delta \) \( \rightarrow d_T(x, y) \approx 2\Delta \). (Level of subtree call.)

\( Pr[\text{cut at level} \Delta] \)?

Would like it to be \( \frac{d(x,y)}{\Delta} \).

\( \rightarrow \) expected length is \( \sum_{\Delta=\frac{D}{2^i}}(2\Delta) \frac{d(x,y)}{\Delta} = 2d(x, y) \).

Why should it be \( \frac{d(x,y)}{\Delta} \)?

smaller the edge the less likely to be on edge of ball.

larger the delta, more room inside ball.

random diameter jiggles edge of ball.

\( \rightarrow Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta \)

The problem?

Could be cut be many different balls.

For each probability is good, but could be hit by many.
Analysis: idea

Claim: $E[d_T(x,y)] = O(\log n) d(x,y)$.
Cut at level $\Delta \rightarrow d_T(x,y) \approx 2\Delta$. (Level of subtree call.)

$Pr[cut \ at \ level\Delta]$?

Would like it to be $\frac{d(x,y)}{\Delta}$.

$\rightarrow$ expected length is $\sum_{\Delta=D/2^i} (2\Delta) \frac{d(x,y)}{\Delta} = 2d(x,y)$.

Why should it be $\frac{d(x,y)}{\Delta}$?

smaller the edge the less likely to be on edge of ball.
larger the delta, more room inside ball.

random diameter jiggles edge of ball.

$\rightarrow Pr[x, y \ cut \ by \ ball| x \ in \ ball] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta$

The problem?

Could be cut be many different balls.
For each probability is good, but could be hit by many.

random permutation to deal with this
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)
(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).
Analysis: \( (x, y) \)

Have \( \Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta \)

(Only consider cut by \( x \), factor 2 loss.)

At level \( \Delta \)

At some point \( x \) is in some \( \Delta \) level ball.
Renumber nodes in order of distance from \( x \).

Can only in ball for \( j \), where \( d(j,x) \in [\Delta/4, \Delta/2] \),
Call this set \( X_\Delta \).

If \( j \in X_\Delta \) cuts \( (x,y) \) if..
Analysis: \((x, y)\)

Have \(\Pr[x, y \text{ cut by ball}| x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Re-number nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if.. \(d(j, x) \leq \beta \Delta\)
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..
\[d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y)\]
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..
\[d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y)\]
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)
(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..

\[
\begin{align*}
  d(j, x) &\leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y) \\
  \Rightarrow \beta \Delta &\in [d[j, x], d(j, x) + d(x, y)].
\end{align*}
\]
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Reenumerate nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if...
\(d(j, x) \leq \beta \Delta\) and \(\beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y)\)
\(\rightarrow \beta \Delta \in [d[j, x], d(j, x) + d(x, y)]\).

occurs with prob. \(\frac{d(x,y)}{\Delta/4} = \frac{4d(x,y)}{\Delta}\).
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta / 4, \Delta / 2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..
\[d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y)\]
\[\rightarrow \beta \Delta \in [d[j, x], d(j, x) + d(x, y)].\]
occurring with prob. \(\frac{d(x, y)}{\Delta / 4} = \frac{4d(x, y)}{\Delta}\).
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j,x) \in [\Delta/4, \Delta/2]\).
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..

\[d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y)\]
\[\rightarrow \beta \Delta \in [d[j, x], d(j, x) + d(x, y)].\]

occurs with prob. \(\frac{d(x,y)}{\Delta/4} = \frac{4d(x,y)}{\Delta}\).

And \(j\) must be before any \(i < j\) in \(\pi\)
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.

Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),

Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..

\[d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y)\]

\[\rightarrow \beta \Delta \in [d[j, x], d(j, x) + d(x, y)].\]

occurs with prob. \(\frac{d(x,y)}{\Delta/4} = \frac{4d(x,y)}{\Delta}\).

And \(j\) must be before any \(i < j\) in \(\pi \rightarrow \text{prob is } \frac{1}{j}\)
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}|x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.

 Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j,x) \in [\Delta/4, \Delta/2]\),

 Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..

\[d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y)\]

\[\rightarrow \beta \Delta \in [d[j, x], d(j, x) + d(x, y)].\]

occurs with prob. \(\frac{d(x,y)}{\Delta/4} = \frac{4d(x,y)}{\Delta}\).

And \(j\) must be before any \(i < j\) in \(\pi\) \(
\rightarrow \text{prob is } \frac{1}{j}
\rightarrow Pr[j \text{ cuts } (x, y)] \leq \left(\frac{1}{j}\right) \frac{4d(x,y)}{\Delta}\)
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}| x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..
\[
\begin{align*}
d(j, x) &\leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y) \\
\rightarrow \beta \Delta &\in [d[j, x], d(j, x) + d(x, y)].
\end{align*}
\]
occurs with prob. \(\frac{d(x,y)}{\Delta/4} = \frac{4d(x,y)}{\Delta}\).

And \(j\) must be before any \(i < j\) in \(\pi\) \(\rightarrow\) prob is \(\frac{1}{j}\)

\(\rightarrow Pr[j \text{ cuts } (x, y)] \leq \left(\frac{1}{j}\right) \frac{4d(x,y)}{\Delta}\)

\(d_T(x,y)\) if cut level \(\Delta\) is \(2\Delta\).
Analysis: \((x, y)\)

Have \(Pr[x, y \text{ cut by ball}| x \text{ in ball}] \approx \frac{d(x, y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).

Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_\Delta\).

If \(j \in X_\Delta\) cuts \((x, y)\) if..
\[
d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y)
\]
\[
\rightarrow \beta \Delta \in [d[j, x], d(j, x) + d(x, y)].
\]
occurs with prob. \(\frac{d(x, y)}{\Delta/4} = \frac{4d(x, y)}{\Delta}\).

And \(j\) must be before any \(i < j\) in \(\pi\) \(\rightarrow\) prob is \(\frac{1}{j}\)

\[
\rightarrow Pr[j \text{ cuts } (x, y)] \leq \left(\frac{1}{j}\right) \frac{4d(x, y)}{\Delta}
\]

\(d_\tau(x, y)\) if cut level \(\Delta\) is \(2\Delta\).
\[
\rightarrow E[d_\tau(x, y)] = \sum_{\Delta=\frac{D}{2^i}} \sum_{j \in X_\Delta} \left(\frac{1}{j}\right) 8d(x, y)
\]
**Analysis: \((x, y)\)**

Have \(Pr[x, y \text{ cut by ball} | x \text{ in ball}] \approx \frac{d(x,y)}{\beta \Delta} \leq 4\Delta\)

(Only consider cut by \(x\), factor 2 loss.)

At level \(\Delta\)

At some point \(x\) is in some \(\Delta\) level ball.
Renumber nodes in order of distance from \(x\).
Can only in ball for \(j\), where \(d(j, x) \in [\Delta/4, \Delta/2]\),
Call this set \(X_{\Delta}\).

If \(j \in X_{\Delta}\) cuts \((x, y)\) if...
\[ d(j, x) \leq \beta \Delta \text{ and } \beta \Delta \leq d(j, y) \leq d(j, x) + d(x, y) \]
\[ \rightarrow \beta \Delta \in [d[j, x], d(j, x) + d(x, y)]. \]
occurs with prob. \(\frac{d(x,y)}{\Delta/4} = \frac{4d(x,y)}{\Delta}\).

And \(j\) must be before any \(i < j\) in \(\pi\) \(\rightarrow\) prob is \(\frac{1}{j}\)

\(\rightarrow Pr[j \text{ cuts } (x, y)] \leq \left(\frac{1}{j}\right) \frac{4d(x,y)}{\Delta}\)

\(d_T(x, y)\) if cut level \(\Delta\) is \(2\Delta\).
\(\rightarrow E[d_T(x, y)] = \sum_{\Delta=\frac{D}{2^l}} \sum_{j \in X_{\Delta}} \left(\frac{1}{j}\right) 8d(x, y)\)
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta = D/2^i} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 2d(x, y) \]
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta = D/2^i} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( \mathcal{X}_\Delta \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \).
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta=D/2} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_\Delta \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

“Listen Stash, the pipes are distinct!!”
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta=D/2} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_{\Delta} \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

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Uh.. well \( X_{\Delta} \) is distinct from \( X_{\Delta/4} \).
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta = D/2^i} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_{\Delta} \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

“Listen Stash, the pipes are distinct!!”

Uh.. well \( X_{\Delta} \) is distinct from \( X_{\Delta/4} \).

\[ E(d_T(x, y)) = \sum_{\Delta = D/4^i} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) + \sum_{\Delta = D/(24^i)} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) \]
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta = D/2^i} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_\Delta \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

“Listen Stash, the pipes are distinct!!”

Uh.. well \( X_\Delta \) is distinct from \( X_{\Delta/4} \).

\[ E(d_T(x, y)) = \sum_{\Delta = \frac{D}{4^i}} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 8d(x, y) + \sum_{\Delta = \frac{D}{((2)^4)^i}} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 8d(x, y) \]
\[ \leq 2 \sum_j \left( \frac{1}{j} \right) 4d(x, y) \]
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta = D/2^i} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_{\Delta} \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

“Listen Stash, the pipes are distinct!!”

Uh.. well \( X_{\Delta} \) is distinct from \( X_{\Delta/4} \).

\[ E(d_T(x, y)) = \sum_{\Delta = D/4^i} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) + \sum_{\Delta = D/(24^i)} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) \]

\[ \leq 2 \sum_{j} \left( \frac{1}{j} \right) 4d(x, y) \]

\[ \leq (16 \ln n) (d(x, y)) \]
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta=D/2^i} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_{\Delta} \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

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Uh.. well \( X_{\Delta} \) is distinct from \( X_{\Delta/4} \).

\[ E(d_T(x, y)) = \sum_{\Delta=\frac{D}{4^i}} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) + \sum_{\Delta=\frac{D}{(2^4)^i}} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) \]

\[ \leq 2 \sum_{j} \left( \frac{1}{j} \right) 4d(x, y) \]

\[ \leq (16\ln n) (d(x, y)) \]

**Claim:** \( E[d_T(x, y)] = O(\log n) d(x, y) \)
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta=D/2^i} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_\Delta \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

“Listen Stash, the pipes are distinct!!”

Uh.. well \( X_\Delta \) is distinct from \( X_{\Delta/4} \).

\[ E(d_T(x, y)) = \sum_{\Delta=D/4^i} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 8d(x, y) + \sum_{\Delta=D/(2^{4i})} \sum_{j \in X_\Delta} \left( \frac{1}{j} \right) 8d(x, y) \]

\[ \leq 2 \sum_{j} \left( \frac{1}{j} \right) 4d(x, y) \]

\[ \leq (16 \ln n) (d(x, y)). \]

**Claim:** \( E[d_T(x, y)] = O(\log n)d(x, y) \)

Expected stretch is \( O(\log n) \).
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta=\frac{D}{2^i}} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_{\Delta} \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

“Listen Stash, the pipes are distinct!!”

Uh.. well \( X_{\Delta} \) is distinct from \( X_{\Delta/4} \).

\[ E(d_T(x, y)) = \sum_{\Delta=\frac{D}{4^i}} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) + \sum_{\Delta=\frac{D}{(2^{2i})}} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) \]

\[ \leq 2 \sum_{j} \left( \frac{1}{j} \right) 4d(x, y) \]

\[ \leq (16 \ln n) (d(x, y)). \]

**Claim:** \( E[d_T(x, y)] = O(\log n) d(x, y) \)

Expected stretch is \( O(\log n) \).

We gave an algorithm that produces a distribution of trees.
The pipes are distinct!

\[ E(d_T(x, y)) = \sum_{\Delta=D/2^i} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 2d(x, y) \]

Recall \( X_{\Delta} \) has nodes with \( d(x, j) \in [\Delta/4, \Delta/2] \)

“Listen Stash, the pipes are distinct!!”

Uh.. well \( X_{\Delta} \) is distinct from \( X_{\Delta/4} \).

\[ E(d_T(x, y)) = \sum_{\Delta=D/2^i} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) + \sum_{\Delta=D/((2)^4i)} \sum_{j \in X_{\Delta}} \left( \frac{1}{j} \right) 8d(x, y) \]
\[ \leq 2 \sum_j \left( \frac{1}{j} \right) 4d(x, y) \]
\[ \leq (16 \ln n)(d(x, y)). \]

**Claim:** \( E[d_T(x, y)] = O(\log n) d(x, y) \)

Expected stretch is \( O(\log n) \).

We gave an algorithm that produces a distribution of trees.

The expected stretch of any pair is \( O(\log n) \).
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \bar{S})}{|S|\bar{|S|}}$, 

Find smallest sparsity cut?

Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.

(Disguised a bit.)

Lemma: $S \geq \sum_{i,j} d(i,j)$.

Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter.

If cut is balanced $|S|\bar{|S|}$ is $\Theta(n^2)$ and sparsity is $O(\log n/D)$.

If not... a bit more work...
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S||\overline{S}|}$,

Find smallest sparsity cut?
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S|\overline{|S|}}$.

Find smallest sparsity cut?

Cheeger: approximately find small expansion cut. (Quadratic approximation.)
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S||\overline{S}|}$,

Find smallest sparsity cut?
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Recall: Expansion estimates sparsity within factor of two.
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S| \overline{|S|}}$, 

Find smallest sparsity cut? 
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

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Toll problem: assign tolls to max. average toll bet. all pairs of vertices.
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S|\overline{|S|}}$.

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

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Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S|\overline{|S|}}$,

Find smallest sparsity cut?

Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem. (Disguised a bit.)

Lemma: $S \geq \frac{1}{\sum_{i,j} d(i,j)}$. 
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S||\overline{S}|}$,

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

Lemma: $\mathcal{I} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \bar{S})}{|S|\bar{S}}$,

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

Lemma: $\mathcal{S} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?
Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter.
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S||\overline{S}|}$,

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

Lemma: $\mathcal{S} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter. $D$ is at least average distance: $\sum_{i,j} d(i,j)/n^2$. 
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S| \overline{|S|}}$.

Find smallest sparsity cut?

Cheeger: approximately find small expansion cut. (Quadratic approximation.)
Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

Lemma: $\mathcal{S} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter. $D$ is at least average distance: $\sum_{i,j} d(i,j)/n^2$.

If cut is balanced
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S||\overline{S}|}$,

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

Lemma: $\mathcal{I} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter. $D$ is at least average distance: $\sum_{i,j} d(i,j)/n^2$.

If cut is balanced $|S||\overline{S}|$ is $\Theta(n^2)$
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \bar{S})}{|S||\bar{S}|}$.

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

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Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter. $D$ is at least average distance: $\sum_{i,j} d(i,j)/n^2$.

If cut is balanced $|S||\bar{S}|$ is $\Theta(n^2)$
and sparsity is $\frac{O(\log n)/D}{cn^2} = \frac{O(\log n)}{\sum_{i,j} d(i,j)}$. 
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S|\overline{|S|}}$, 

Find smallest sparsity cut?  
Cheeger: approximately find small expansion cut. (Quadratic approximation.)
Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.  
(Disguised a bit.)

Lemma: $\mathcal{L} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter.  
$D$ is at least average distance: $\sum_{i,j} d(i,j)/n^2$.

If cut is balanced $|S|\overline{|S|}$ is $\Theta(n^2)$
and sparsity is $\frac{O(\log n)/D}{cn^2} = \frac{O(\log n)}{\sum_{i,j} d(i,j)}$.

→ find $O(\log n)$ times optimal sparse cut.
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, S)}{|S|S}$,

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.

Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

Lemma: $\mathcal{S} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter. $D$ is at least average distance: $\sum_{i,j} d(i,j)/n^2$.

If cut is balanced $|S||\overline{S}|$ is $\Theta(n^2)$
and sparsity is $\frac{O(\log n)/D}{cn^2} = \frac{O(\log n)}{\sum_{i,j} d(i,j)}$.

$\rightarrow$ find $O(\log n)$ times optimal sparse cut.

If not...
Alternative to Cheeger for expansion.

Graph $G$, sparsity of cut $\frac{E(S, \overline{S})}{|S||\overline{S}|}$,

Find smallest sparsity cut?
Cheeger: approximately find small expansion cut. (Quadratic approximation.)

Recall: Expansion estimates sparsity within factor of two.

Toll problem: assign tolls to max. average toll bet. all pairs of vertices.
Exam: Sparsity of graph is lower bounded by function of toll problem.
(Disguised a bit.)

Lemma: $\\mathcal{S} \geq \frac{1}{\sum_{i,j} d(i,j)}$.

Given solution to toll problem, find cut?

Top level cuts each edge with prob. $O(\log n)/D$, $D$ is diameter. $D$ is at least average distance: $\sum_{i,j} d(i,j) / n^2$.

If cut is balanced $|S||\overline{S}|$ is $\Theta(n^2)$

and sparsity is $\frac{O(\log n)/D}{cn^2} = \frac{O(\log n)}{\sum_{i,j} d(i,j)}$.

→ find $O(\log n)$ times optimal sparse cut.

If not...a bit more work...
Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Idea: find HST for metric $(X, d)$. Solve the problem on a hierarchically well separated tree metric. Kleinberg-Tardos: constant factor on uniform metric. Hierarchically well separated tree, "geometric", constant factor. $\Rightarrow O(\log n)$ approximation.
Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \rightarrow X$ that minimizes
Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \to X$ that minimizes

$$\sum_{e=(u,v)} c(e) d(l(u), l(v)) + \sum_v c(v, l(v))$$
Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \to X$ that minimizes

$$\sum_{e=(u,v)} c(e) d(l(u), l(v)) + \sum_{v} c(v, l(v))$$

Idea: find HST for metric $(X, d)$. 
Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \rightarrow X$ that minimizes

$$\sum_{e=(u,v)} c(e)d(\ell(u), \ell(v)) + \sum_v c(v, \ell(v))$$

Idea: find HST for metric $(X, d)$.

Solve the problem on a hierarchically well separated tree metric.
Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \rightarrow X$ that minimizes

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Idea: find HST for metric $(X, d)$.

Solve the problem on a hierarchically well separated tree metric.

Kleinberg-Tardos: constant factor on uniform metric.
Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \to X$ that minimizes

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Idea: find HST for metric $(X, d)$.

Solve the problem on a hierarchically well separated tree metric.

Kleinberg-Tardos: constant factor on uniform metric.

Hierarchically well separated tree,
Metric Labelling

Input: graph \(G = (V, E)\) with edge weights, \(w(\cdot)\), metric labels \((X, d)\), and costs for mapping vertices to labels \(c : V \times X\).

Find an labeling of vertices, \(\ell : V \to X\) that minimizes

\[
\sum_{e=(u,v)} c(e) d(\ell(u), \ell(v)) + \sum_v c(v, \ell(v))
\]

Idea: find HST for metric \((X, d)\).

Solve the problem on a hierarchically well separated tree metric.

Kleinberg-Tardos: constant factor on uniform metric.

Hierarchically well separated tree, “geometric”,

Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

Find an labeling of vertices, $\ell : V \rightarrow X$ that minimizes

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Idea: find HST for metric $(X, d)$.

Solve the problem on a hierarchically well separated tree metric.

Kleinberg-Tardos: constant factor on uniform metric.

Hierarchically well separated tree, “geometric”, constant factor.
Metric Labelling

Input: graph $G = (V, E)$ with edge weights, $w(\cdot)$, metric labels $(X, d)$, and costs for mapping vertices to labels $c : V \times X$.

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$$\sum_{e=(u,v)} c(e) d(\ell(u), \ell(v)) + \sum_v c(v, \ell(v))$$

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Solve the problem on a hierarchically well separated tree metric.

Kleinberg-Tardos: constant factor on uniform metric.

Hierarchically well separated tree, “geometric”, constant factor.

$\rightarrow O(\log n)$ approximation.
See you ...

Tuesday.