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Review: Spectral gap, Edge expansion $h(G)$, Sparsity $\phi(G)$ etc.
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Write $1 - \lambda_2$ as a relaxation of $\phi(G)$, Cheeger easy part
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Cheeger hard part: Sweeping cut Algorithm, Proof, Asymptotic tight example
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Cheeger hard part: Sweeping cut Algorithm, Proof, Asymptotic tight example
Graph $G = (V, E)$,
Edge Expansion/Conductance.

Graph $G = (V, E)$,

Assume regular graph of degree $d$. 

Note $n \geq \max(|S|, |V| - |S|) \geq n/2$.

$\Rightarrow h(G) \leq \phi(G) \leq 2h(S)$. 


Edge Expansion/Conductance.

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Edge Expansion.
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**Edge Expansion.**

$$h(S) = \frac{|E(S, V - S)|}{d \min(|S|, |V - S|)}$$  \hspace{1cm} h(G) = \min_{S \subset V} h(S)$$
Edge Expansion/Conductance.

Graph \( G = (V, E) \),

Assume regular graph of degree \( d \).

Edge Expansion.

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h(S) = \frac{|E(S, V - S)|}{d \min(|S|, |V - S|)}, \quad h(G) = \min_{S \subset V} h(S)
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Conductance (Sparsity).

\[
\phi(S) = \frac{|E(S, V - S)|}{d |S| |V - S|}, \quad \phi(G) = \min_{S \subset V} \phi(S)
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\( n \geq \max(|S|, |V - S|) \geq n/2 \) → \( h(G) \leq \phi(G) \leq 2h(S) \)
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$$h(S) = \frac{|E(S, V - S)|}{d \min(|S|, |V - S|)}, \quad h(G) = \min_{S \subseteq V} h(S)$$

**Conductance (Sparsity).**

$$\phi(S) = \frac{n|E(S, V - S)|}{d|S||V - S|}, \quad \phi(G) = \min_{S \subseteq V} \phi(S)$$
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Spectra of the graph.

\[ A \text{: Adjacency Matrix } A_{ij} = 1 \iff (i, j) \in E \]
Spectra of the graph.

$A$: Adjacency Matrix $A_{ij} = 1 \iff (i, j) \in E$

$M = \frac{1}{d} A$, normalized adjacency matrix,
Spectra of the graph.

\[ A: \text{Adjacency Matrix } A_{ij} = 1 \iff (i, j) \in E \]
\[ M = \frac{1}{d} A, \text{ normalized adjacency matrix, } M \text{ real, symmetric} \]
Spectra of the graph.

$A$: Adjacency Matrix $A_{ij} = 1 \Leftrightarrow (i, j) \in E$

$M = \frac{1}{d}A$, normalized adjacency matrix, $M$ real, symmetric

orthonormal eigenvectors: $v_1, \ldots, v_n$ with eigenvalues

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$
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Claim: Any two eigenvectors with different eigenvalues are orthogonal.
Spectra of the graph.

\[ A: \text{Adjacency Matrix } A_{ij} = 1 \iff (i, j) \in E \]

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orthonormal eigenvectors: \( \nu_1, \ldots, \nu_n \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \)

**Claim:** Any two eigenvectors with different eigenvalues are orthogonal.

**Proof:** Eigenvectors: \( \nu, \nu' \) with eigenvalues \( \lambda, \lambda' \).
Spectra of the graph.

$A$: Adjacency Matrix $A_{ij} = 1 \iff (i, j) \in E$

$M = \frac{1}{d}A$, normalized adjacency matrix, $M$ real, symmetric orthonormal eigenvectors: $v_1, \ldots, v_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

Claim: Any two eigenvectors with different eigenvalues are orthogonal.

Proof: Eigenvectors: $v, v'$ with eigenvalues $\lambda, \lambda'$.

$v^T M v' = v^T (\lambda' v')$
Spectra of the graph.

A: Adjacency Matrix \( A_{ij} = 1 \iff (i, j) \in E \)

\( M = \frac{1}{d} A \), normalized adjacency matrix, \( M \) real, symmetric

orthonormal eigenvectors: \( v_1, \ldots, v_n \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \)

**Claim:** Any two eigenvectors with different eigenvalues are orthogonal.

**Proof:** Eigenvectors: \( v, v' \) with eigenvalues \( \lambda, \lambda' \).

\[
\begin{align*}
v^T M v' &= v^T (\lambda' v') = \lambda' v^T v'
\end{align*}
\]
Spectra of the graph.

\[ A: \text{Adjacency Matrix } A_{ij} = 1 \iff (i,j) \in E \]

\[ M = \frac{1}{d} A, \text{ normalized adjacency matrix, } M \text{ real, symmetric} \]

Orthonormal eigenvectors: \( v_1, \ldots, v_n \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \)

**Claim:** Any two eigenvectors with different eigenvalues are orthogonal.

**Proof:** Eigenvectors: \( v, v' \) with eigenvalues \( \lambda, \lambda' \).

\[ v^T M v' = v^T (\lambda' v') = \lambda' v^T v' \]

\[ v^T M v' = \lambda v^T v' \]
Spectra of the graph.

A: Adjacency Matrix $A_{ij} = 1 \iff (i, j) \in E$

$M = \frac{1}{d} A$, normalized adjacency matrix, $M$ real, symmetric

orthonormal eigenvectors: $v_1, \ldots, v_n$ with eigenvalues

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

Claim: Any two eigenvectors with different eigenvalues are orthogonal.

Proof: Eigenvectors: $v, v'$ with eigenvalues $\lambda, \lambda'$.

$v^T M v' = v^T (\lambda' v') = \lambda' v^T v'$

$v^T M v' = \lambda v^T v' = \lambda v^T v$. 

□
Action of $M$.

$\nu$ - assigns values to vertices.
Action of $M$.

$v$ - assigns values to vertices.

$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$
Action of $M$.

$v$ - assigns values to vertices.

$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$  

Action of $M$: taking the average of your neighbours.
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(Direct) result from the action of $M$,
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Action of $M$: taking the average of your neighbours.

(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$
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(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$

$v_1 = 1.$
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$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$  

Action of $M$: taking the average of your neighbours.

(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$

$v_1 = 1$. $\lambda_1 = 1$.

**Claim:** For a connected graph $\lambda_2 < 1$. 
**Action of $M$.**

$v$ - assigns values to vertices.

$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$  

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$v_1 = 1$. $\lambda_1 = 1$.

**Claim:** For a connected graph $\lambda_2 < 1$.

**Proof:** Second Eigenvector: $v \perp 1$. 
Action of $M$.

$v$ - assigns values to vertices.

$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$  

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$v_1 = 1$. $\lambda_1 = 1$.

Claim: For a connected graph $\lambda_2 < 1$.

Proof: Second Eigenvector: $v \perp 1$. Max value $x$. Connected $\rightarrow$ path from $x$ valued node to lower value.
Action of $M$.

$v$ - assigns values to vertices.

$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$  

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**Claim:** For a connected graph $\lambda_2 < 1$.

**Proof:** Second Eigenvector: $v \perp 1$. Max value $x$.

Connected $\rightarrow$ path from $x$ valued node to lower value.

$\rightarrow \exists e = (i,j), \; v_i = x, \; x_j < x.$
Action of $M$. 

$v$ - assigns values to vertices.

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Connected $\rightarrow$ path from $x$ valued node to lower value.

$\rightarrow \exists e = (i, j), v_i = x, x_j < x.$

\[ \frac{i}{j} \]

$\vdots$

\[ \frac{x}{x} \leq x \]
Action of $M$.

$\nu$ - assigns values to vertices.

$$(M\nu)_i = \frac{1}{d} \sum_{j \sim i} \nu_j.$$  

Action of $M$: taking the average of your neighbours.

(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$

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**Proof:** Second Eigenvector: $\nu \perp 1$. Max value $x$.

Connected $\rightarrow$ path from $x$ valued node to lower value.

$\therefore \exists e = (i, j), \nu_i = x, x_j < x.$

$$(M\nu)_i \leq \frac{1}{d} (x + x \cdots + v_j) \leq x$$
Action of $M$.

$v$ - assigns values to vertices.

$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$  

Action of $M$: taking the average of your neighbours.

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$(Mv)_i \leq \frac{1}{d} (x + x \cdots + v_j) < x.$
Action of $M$.

- Assigns values to vertices.

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Connected $\rightarrow$ path from $x$ valued node to lower value.

$\exists e = (i,j)$, $v_i = x$, $x_j < x$.

$$(Mv)_i \leq \frac{1}{d} (x + x \cdots + v_j) < x.$$  

Therefore $\lambda_2 < 1$. 

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$v$ - assigns values to vertices.

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Connected $\rightarrow$ path from $x$ valued node to lower value.

$\rightarrow \exists e = (i,j), v_i = x, x_j < x$. 

![Diagram](image)

$$(Mv)_i \leq \frac{1}{d}(x + x \cdots + v_j) < x.$$ 

Therefore $\lambda_2 < 1$.  

\[\square\]
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**Claim:** For a connected graph $\lambda_2 < 1$.

**Proof:** Second Eigenvector: $v \perp 1$. Max value $x$.
Connected $\rightarrow$ path from $x$ valued node to lower value.  

$\exists e = (i, j), v_i = x, x_j < x.$

$$(Mv)_i \leq \frac{1}{d} (x + x \cdots + v_j) < x.$$  

Therefore $\lambda_2 < 1$.  

**Claim:** Connected if $\lambda_2 < 1$.  

Action of $M$.

- Assigns values to vertices.

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Action of $M$: taking the average of your neighbours.

(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$

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**Proof:** Second Eigenvector: $v \perp 1$. Max value $x$.

Connected $\rightarrow$ path from $x$ valued node to lower value.

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$$(Mv)_i \leq \frac{1}{d} (x + x \cdots + v_j) < x.$$  

Therefore $\lambda_2 < 1$.  

**Claim:** Connected if $\lambda_2 < 1$.

**Proof:** By contradiction. Assign $+1$ to vertices in one component, $-\delta$ to rest.
Action of $M$.

- $\nu$ - assigns values to vertices.

$$(M\nu)_i = \frac{1}{d} \sum_{j \sim i} \nu_j.$$ 

Action of $M$: taking the average of your neighbours.

(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$

$\nu_1 = 1. \quad \lambda_1 = 1.$

Claim: For a connected graph $\lambda_2 < 1$.

Proof: Second Eigenvector: $\nu \perp 1$. Max value $x$.
Connected $\rightarrow$ path from $x$ valued node to lower value.

$\rightarrow \exists \ e = (i,j), \ \nu_i = x, \ x_j < x.$

\[
\begin{align*}
\begin{array}{c}
\text{i} \\
\text{x}
\end{array} \
\begin{array}{c}
\text{j} \\
\leq x
\end{array}
\end{align*}
\]

\[
(M\nu)_i \leq \frac{1}{d}(x + x \cdots + \nu_j) < x.
\]

Therefore $\lambda_2 < 1. \quad \Box$

Claim: Connected if $\lambda_2 < 1$.

Proof: By contradiction. Assign $+1$ to vertices in one component, $-\delta$ to rest.

$\ x_i = (Mx_i)$
Action of \( M \).

\( \nu \) - assigns values to vertices.

\[
(M\nu)_i = \frac{1}{d} \sum_{j \sim i} \nu_j.
\]

Action of \( M \): taking the average of your neighbours.

(Direct) result from the action of \( M \), \( |\lambda_i| \leq 1 \) \( \forall i \)

\( \nu_1 = 1. \lambda_1 = 1. \)

Claim: For a connected graph \( \lambda_2 < 1. \)

Proof: Second Eigenvector: \( \nu \perp 1 \). Max value \( x \).
Connected \( \implies \) path from \( x \) valued node to lower value.

\[ \exists e = (i, j), \nu_i = x, x_j < x. \]

\[ x \leq x \]

\[ (M\nu)_i \leq \frac{1}{d} (x + x \cdots + \nu_j) < x. \]

Therefore \( \lambda_2 < 1. \) \( \square \)

Claim: Connected if \( \lambda_2 < 1. \)

Proof: By contradiction. Assign \( +1 \) to vertices in one component, \( -\delta \) to rest.

\( x_i = (Mx_i) \implies \) eigenvector with \( \lambda = 1. \)
Action of $M$.

$v$ - assigns values to vertices.

$$(Mv)_i = \frac{1}{d} \sum_{j \sim i} v_j.$$  

Action of $M$: taking the average of your neighbours.

(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$

$v_1 = 1. \quad \lambda_1 = 1.$

Claim: For a connected graph $\lambda_2 < 1$.

Proof: Second Eigenvector: $v \perp 1$. Max value $x$.
Connected $\rightarrow$ path from $x$ valued node to lower value.

$\rightarrow \exists e = (i, j), v_i = x, x_j < x.$

\[ (Mv)_i \leq \frac{1}{d} (x + x \cdots + v_j) < x. \]

Therefore $\lambda_2 < 1.$

Claim: Connected if $\lambda_2 < 1$.

Proof: By contradiction. Assign $+1$ to vertices in one component, $-\delta$ to rest.

$x_i = (Mx_i) \implies$ eigenvector with $\lambda = 1$.

Choose $\delta$ to make $\sum_i x_i = 0$, 

Action of $M$.

$v$ - assigns values to vertices.

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Action of $M$: taking the average of your neighbours.

(Direct) result from the action of $M$, $|\lambda_i| \leq 1 \quad \forall i$

$v_1 = 1$. $\lambda_1 = 1$.

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**Proof**: Second Eigenvector: $v \perp 1$. Max value $x$.

Connected $\rightarrow$ path from $x$ valued node to lower value.

$\rightarrow \exists \ e = (i, j), v_i = x, x_j < x.$

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\leq x
\end{array} \\
\begin{array}{c}
i \\
\downarrow
\end{array} \\
\begin{array}{c}
x \\
\downarrow
\end{array} \\
\begin{array}{c}
j
\end{array}
\end{array}
\]

\[
(Mv)_i \leq \frac{1}{d}(x + x \cdots + v_j) < x.
\]

Therefore $\lambda_2 < 1$. 

**Claim**: Connected if $\lambda_2 < 1$.

**Proof**: By contradiction. Assign $+1$ to vertices in one component, $-\delta$ to rest.

$x_i = (Mx_i) \implies \text{eigenvector with } \lambda = 1.$

Choose $\delta$ to make $\sum_i x_i = 0$, i.e., $x \perp 1$. 

\[\square\]
Spectral Gap and the connectivity of graph.

Spectral gap: $\mu = \lambda_1 - \lambda_2 = 1 - \lambda_2$. 
Spectral Gap and the connectivity of graph.

Spectral gap: \( \mu = \lambda_1 - \lambda_2 = 1 - \lambda_2 \).

Recall: \( h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V - S)|}{|S|} \)
Spectral Gap and the connectivity of graph.

Spectral gap: \( \mu = \lambda_1 - \lambda_2 = 1 - \lambda_2. \)

Recall: \( h(G) = \min_{S, |S| < |V|/2} \frac{|E(S, V - S)|}{|S|} \)

\( 1 - \lambda_2 = 0 \iff \lambda_2 = 1 \iff G \text{ disconnected} \iff h(G) = 0 \)
Spectral Gap and the connectivity of graph.

Spectral gap: $\mu = \lambda_1 - \lambda_2 = 1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$1 - \lambda_2 = 0 \iff \lambda_2 = 1 \iff G$ disconnected $\iff h(G) = 0$

In general, small spectral gap $1 - \lambda_2$ suggests ”poorly connected” graph.
Spectral Gap and the connectivity of graph.

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Formally
Spectral Gap and the connectivity of graph.

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Formally

**Cheeger’s Inequality**
Spectral Gap and the connectivity of graph.

Spectral gap: $\mu = \lambda_1 - \lambda_2 = 1 - \lambda_2$.

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$1 - \lambda_2 = 0 \iff \lambda_2 = 1 \iff G$ disconnected $\iff h(G) = 0$

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Formally

**Cheeger’s Inequality**

$$\frac{1 - \lambda_2}{2}$$
Spectral Gap and the connectivity of graph.

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\( 1 - \lambda_2 = 0 \iff \lambda_2 = 1 \iff G \text{ disconnected} \iff h(G) = 0 \)

In general, small spectral gap \( 1 - \lambda_2 \) suggests "poorly connected" graph

Formally

**Cheeger's Inequality**

\[
\frac{1 - \lambda_2}{2} \leq h(G)
\]
Spectral Gap and the connectivity of graph.

Spectral gap: $\mu = \lambda_1 - \lambda_2 = 1 - \lambda_2$.

Recall: $h(G) = \min_{S, |S| \leq |V|/2} \frac{|E(S, V-S)|}{|S|}$

$1 - \lambda_2 = 0 \iff \lambda_2 = 1 \iff G \text{ disconnected} \iff h(G) = 0$

In general, small spectral gap $1 - \lambda_2$ suggests "poorly connected" graph.

Formally

**Cheeger’s Inequality**

$$\frac{1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)}$$
Spectral Gap and Conductance.

We will show $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$. 
Spectral Gap and Conductance.

We will show $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$.

$$\phi(G) = \min_{S \in V} \frac{n |E(S, V - S)|}{d |S| |V - S|}$$
Spectral Gap and Conductance.

We will show $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$.

$$\phi(G) = \min_{S \in V} \frac{n|E(S, V - S)|}{d|S||V - S|}$$

Let $x$ be the characteristic vector of set $S$. 

Spectral Gap and Conductance.

We will show $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$.

$$\phi(G) = \min_{S \in V} \frac{n|E(S, V - S)|}{d|S||V - S|}$$

Let $x$ be the characteristic vector of set $S$  
$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$
Spectral Gap and Conductance.

We will show $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$.

$$\phi(G) = \min_{S \subseteq V} \frac{n|E(S, V - S)|}{d|S||V - S|}$$

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$$|S| |V - S| = \frac{1}{2} \sum_{i,j} |x_i - x_j| = \frac{1}{2} \sum_{i,j} (x_i - x_j)^2$$
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$$\phi(G) = \min_{x \in \{0,1\}^V - \{0,1\}} \frac{n \sum_{i,j} M_{ij} (x_i - x_j)^2}{\sum_{i,j} (x_i - x_j)^2}$$
Recall Rayleigh Quotient: \( \lambda_2 = \max_{x \in \mathbb{R}^n \setminus \{0\}, \perp 1} \frac{x^T M x}{x^T x} \)
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**Claim:** \( 2x^T x = \frac{1}{n} \sum_{i,j} (x_i - x_j)^2 \)
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Claim: $2x^T x = \frac{1}{n} \sum_{i,j} (x_i - x_j)^2$

Proof:

$$\sum_{i,j} (x_i - x_j)^2 = \sum_{i,j} x_i^2 + x_j^2 - 2x_i x_j$$

$$= 2n \sum_i x_i^2 - 2(\sum_i x_i)^2 = 2n \sum_i x_i^2 = 2nx^T x$$

We used $x \perp 1 \Rightarrow \sum_i x_i = 0$
Recall Rayleigh Quotient: $\lambda_2 = \max_{x \in \mathbb{R}^V - \{0\}, x \perp 1} \frac{x^T M x}{x^T x}$

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**Claim:** $2(x^T x - x^T M x) = \sum_{i,j} M_{ij} (x_i - x_j)^2$

**Proof:**

$$\sum_{i,j} M_{ij} (x_i - x_j)^2 = \sum_{i,j} M_{ij} (x_i^2 + x_j^2) - 2 \sum_{i,j} M_{ij} x_i x_j$$

$$= \sum_{i} \sum_{j \sim i} \frac{1}{d} (x_i^2 + x_j^2) - 2x^T M x$$

$$= 2 \sum_{(i,j) \in E} \frac{1}{d} (x_i^2 + x_j^2) - 2x^T M x$$

$$= 2 \sum_{i} x_i^2 - 2x^T M x = 2x^T x - 2x^T M x$$
Combining the two claims, we get
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\[ = \min_{x \in \mathbb{R}^V - \text{Span}\{1\}} \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2} \]
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Recall

\[ \phi(G) = \min_{x \in \{0,1\}^V \setminus \{0,1\}} \frac{n \sum_{i,j} M_{ij}(x_i - x_j)^2}{\sum_{i,j} (x_i - x_j)^2} \]
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\[ 1 - \lambda_2 \leq \phi(G) \]
Combining the two claims, we get

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\[ 1 - \lambda_2 \leq \phi(G) \leq 2h(G) \]
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\[ 1 - \lambda_2 \leq \phi(G) \leq 2h(G) \]

Hooray!! We get the easy part of Cheeger \(\frac{1 - \lambda_2}{2} \leq h(G)\)
Now let’s get to the hard part of Cheeger $h(G) \leq \sqrt{2(1 - \lambda_2)}$. 
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**Idea:** We have $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$. 
Cheeger Hard Part.

Now let’s get to the hard part of Cheeger $h(G) \leq \sqrt{2(1 - \lambda_2)}$.

Idea: We have $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$

Take the 2nd eigenvector $x = \arg\min_{x \in \mathbb{R}^V - \text{Span}\{1\}} \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$
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Now let’s get to the hard part of Cheeger $h(G) \leq \sqrt{2(1 - \lambda_2)}$.

**Idea**: We have $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$

Take the $2^{nd}$ eigenvector $x = \arg\min_{x \in \mathbb{R}^V - \text{Span}\{1\}} \frac{\sum_{i,j} M_{ij}(x_i-x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i-x_j)^2}$

Consider $x$ as an embedding of the vertices to the real line.
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Round $x$ to get a $x \in \{0, 1\}^V$
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Take the $2^{nd}$ eigenvector $x = \text{argmin}_{x \in \mathbb{R}^V - \text{Span}\{1\}} \frac{1}{n} \sum_{i,j} (x_i - x_j)^2 M_{ij}$

Consider $x$ as an embedding of the vertices to the real line.

Round $x$ to get a $x \in \{0, 1\}^V$

**Rounding:** Take a threshold $t$,

\[
\begin{align*}
  x_i &\geq t \quad \rightarrow x_i = 1 \\
 x_i &< t \quad \rightarrow x_i = 0
\end{align*}
\]
Now let’s get to the hard part of Cheeger $h(G) \leq \sqrt{2(1 - \lambda_2)}$.

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What will be a good $t$?
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\]

What will be a good $t$?

We don’t know. Try all possible thresholds ($n - 1$ possibilities), and hope there is a $t$ leading to a good cut!
Sweeping Cut Algorithm

Input: \( G = (V, E), x \in \mathbb{R}^V, x \perp 1 \)
Sweeping Cut Algorithm

Input: $G = (V, E), x \in \mathbb{R}^V, x \perp \mathbf{1}$

Sort the vertices in non-decreasing order in terms of their values in $x$

WLOG $V = \{1, \ldots, n\}$ \quad $x_1 \leq x_2 \leq \ldots \leq x_n$
Sweeping Cut Algorithm

Input: \( G = (V, E), \ x \in \mathbb{R}^V, x \perp 1 \)

Sort the vertices in non-decreasing order in terms of their values in \( x \)

WLOG \( V = \{1, \ldots, n\} \quad x_1 \leq x_2 \leq \ldots \leq x_n \)

Let \( S_i = \{1, \ldots, i\} \quad i = 1, \ldots, n - 1 \)
Sweeping Cut Algorithm

Input: $G = (V, E)$, $x \in \mathbb{R}^V$, $x \perp 1$

Sort the vertices in non-decreasing order in terms of their values in $x$

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Return $S = \text{argmin}_{S_i} h(S_i)$
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**Main Lemma:** $G = (V, E), d$-regular

$x \in \mathbb{R}^V, x \perp 1, \delta = \frac{\sum_{i,j} M_{ij}(x_i-x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i-x_j)^2}$

If $S$ is the output of the sweeping cut algorithm, then $h(S) \leq \sqrt{2\delta}$
Sweeping Cut Algorithm

Input: $G = (V, E), x \in \mathbb{R}^V, x \perp 1$

Sort the vertices in non-decreasing order in terms of their values in $x$
WLOG $V = \{1, \ldots, n\}$ \quad $x_1 \leq x_2 \leq \ldots \leq x_n$

Let $S_i = \{1, \ldots, i\} \quad i = 1, \ldots, n - 1$

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**Main Lemma:** $G = (V, E)$, $d$-regular
$x \in \mathbb{R}^V, x \perp 1, \delta = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n} \sum_{i,j}(x_i - x_j)^2}$

If $S$ is the output of the sweeping cut algorithm, then $h(S) \leq \sqrt{2\delta}$

**Note:** Applying the Main Lemma with the 2nd eigenvector $v_2$, we have $\delta = 1 - \lambda_2$, and $h(G) \leq h(S) \leq \sqrt{2(1 - \lambda_2)}$. Done!
Proof of Main Lemma

WLOG $V = \{1, \ldots, n\}$ \quad $x_1 \leq x_2 \leq \ldots \leq x_n$
Proof of Main Lemma

WLOG $V = \{1, \ldots, n\}$ \hspace{1cm} $x_1 \leq x_2 \leq \ldots \leq x_n$

Want to show

$$\exists i \text{ s.t. } h(S_i) = \frac{1}{d} \frac{|E(S, V - S)|}{\min(|S|, |V - S|)} \leq \sqrt{2\delta}$$
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$$\exists i \text{ s.t. } h(S_i) = \frac{\frac{1}{d}|E(S, V - S)|}{\min(|S|, |V - S|)} \leq \sqrt{2\delta}$$

Probabilistic Argument: Construct a distribution $D$ over $\{S_1, \ldots, S_{n-1}\}$ such that

$$\mathbb{E}_{S \sim D}[\frac{\frac{1}{d}|E(S, V - S)|}{\mathbb{E}_{S \sim D}[\min(|S|, |V - S|)]}] \leq \sqrt{2\delta}$$
Proof of Main Lemma

WLOG \( V = \{1, \ldots, n\} \) \( x_1 \leq x_2 \leq \ldots \leq x_n \)

Want to show

\[ \exists i \text{ s.t. } h(S_i) = \frac{\frac{1}{d} |E(S, V - S)|}{\min(|S|, |V - S|)} \leq \sqrt{2\delta} \]

**Probabilistic Argument**: Construct a distribution \( D \) over \( \{S_1, \ldots, S_{n-1}\} \) such that

\[ \mathbb{E}_{S \sim D}[\frac{\frac{1}{d} |E(S, V - S)|}{\min(|S|, |V - S|)}] \leq \sqrt{2\delta} \]

\[ \to \mathbb{E}_{S \sim D}[\frac{1}{d} |E(S, V - S)| - \sqrt{2\delta} \min(|S|, |V - S|)] \leq 0 \]
Proof of Main Lemma

WLOG \( V = \{1, \ldots, n\} \quad x_1 \leq x_2 \leq \ldots \leq x_n \)

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**Probabilistic Argument**: Construct a distribution \( D \) over \( \{S_1, \ldots, S_{n-1}\} \) such that

\[ \mathbb{E}_{S \sim D}[\frac{1}{d} |E(S, V - S)|] \leq \sqrt{2\delta} \]

\[ \mathbb{E}_{S \sim D}[\min(|S|, |V - S|)] \]

\[ \rightarrow \mathbb{E}_{S \sim D}[\frac{1}{d} |E(S, V - S)| - \sqrt{2\delta} \min(|S|, |V - S|)] \leq 0 \]

\[ \exists S \quad \frac{1}{d} |E(S, V - S)| - \sqrt{2\delta} \min(|S|, |V - S|) \leq 0 \]
The distribution $D$

WLOG, shift and scale so that $x_{\left\lfloor \frac{n}{2} \right\rfloor} = 0$, and $x_1^2 + x_n^2 = 1$
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Take $t$ from the range $[x_1, x_n]$ with density function $f(t) = 2|t|$.

Check: $\int_{x_1}^{x_n} f(t) dt = \int_{x_1}^0 -2tdt + \int_0^{x_n} 2tdt = x_1^2 + x_n^2 = 1$
The distribution $D$

WLOG, shift and scale so that $x_{\left\lfloor \frac{n}{2} \right\rfloor} = 0$, and $x_1^2 + x_n^2 = 1$

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$S = \{i : x_i \leq t\}$
The distribution \( D \)

WLOG, shift and scale so that \( x_{\lceil n/2 \rceil} = 0 \), and \( x_1^2 + x_n^2 = 1 \)

Take \( t \) from the range \([x_1, x_n]\) with density function \( f(t) = 2|t| \).

Check: \( \int_{x_1}^{x_n} f(t) \, dt = \int_{x_1}^{0} -2t \, dt + \int_{0}^{x_n} 2t \, dt = x_1^2 + x_n^2 = 1 \)

\( S = \{ i : x_i \leq t \} \)

Take \( D \) as the distribution over \( S_1, \ldots, S_{n-1} \) resulted from the above procedure.
Goal: \[ \frac{\mathbb{E}_{S \sim D} \left[ \frac{1}{d} |E(S, V-S)| \right]}{\mathbb{E}_{S \sim D} \left[ \min(|S|, |V-S|) \right]} \leq \sqrt{2 \delta} \]
Goal: \( \frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S \sim D}[\min(|S|,|V-S|)]} \leq \sqrt{2\delta} \)

**Denominator:**
Goal: \( \frac{\mathbb{E}_{S \sim D}[\frac{1}{d} |E(S, V-S)|]}{\mathbb{E}_{S \sim D}[\min(|S|, |V-S|)]} \leq \sqrt{2\delta} \)

**Denominator:**

Let \( T_i = i \) is in the smaller set of \( S, V - S \)
Goal: \[ \frac{\mathbb{E}_{S \sim D} \left[ \frac{1}{d} |E(S, V - S)| \right]}{\mathbb{E}_{S \sim D} \left[ \min(|S|, |V - S|) \right]} \leq \sqrt{2\delta} \]

Denominator:

Let \( T_i = i \) is in the smaller set of \( S, V - S \)

Can check

\[ \mathbb{E}_{S \sim D}[T_i] = Pr[T_i] = x_i^2 \]

\[ \mathbb{E}_{S \sim D}[\min(|S|, |V - S|)] = \mathbb{E}_{S \sim D}[\sum_i T_i] \]

\[ = \sum_i \mathbb{E}_{S \sim D}[T_i] \]

\[ = \sum_i x_i^2 \]
Goal: $$\frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V-S)|]}{\mathbb{E}_{S \sim D}[\min(|S|, |V-S|)]} \leq \sqrt{2\delta}$$
Goal: \[
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\]

**Numerator:**
Goal: \[ \frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V - S)|]}{\mathbb{E}_{S \sim D}[\min(|S|, |V - S|)]} \leq \sqrt{2\delta} \]

**Numerator:**

Let \( T_{i,j} = i, j \) is cut by \((S, V - S)\)
Goal: \( \frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V-S)|]}{\mathbb{E}_{S \sim D}[\min(|S|, |V-S|)]]} \leq \sqrt{2\delta} \)

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Let \( T_{i,j} = i, j \) is cut by \((S, V - S)\)

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\begin{align*}
\begin{cases}
  x_i, x_j \text{ same sign:} & \quad Pr[T_{i,j}] = |x_i^2 - x_j^2| \\
  x_i, x_j \text{ different sign:} & \quad Pr[T_{i,j}] = x_i^2 + x_j^2
\end{cases}
\end{align*}
\]
Goal: \[ \frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S \sim D}[\text{min}(|S|,|V-S|)]} \leq \sqrt{2\delta} \]

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A common upper bound: \( \mathbb{E}[T_{i,j}] = Pr[T_{i,j}] \leq |x_i - x_j|(|x_i| + |x_j|) \)
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\[
\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S, V-S)|] = \frac{1}{2} \sum_{i,j} M_{ij} \mathbb{E}[T_{i,j}] \leq \frac{1}{2} \sum_{i,j} M_{ij} |x_i - x_j|(|x_i| + |x_j|)
\]
Cauchy-Schwarz Inequality

\[ |a \cdot b| \leq \|a\|\|b\|, \text{ as } a \cdot b = \|a\|\|b\| \cos(a, b) \]
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Applying with \( a, b \in \mathbb{R}^n \) with 
\( a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}|x_i| + |x_j| \)
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Applying with \( a, b \in \mathbb{R}^n \) with \( a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}|x_i| + |x_j| \)

**Numerator:**

\[
\mathbb{E}_{S \sim D} \left[ \frac{1}{d} |E(S, V - S)| \right] = \frac{1}{2} \sum_{i,j} M_{ij} \mathbb{E}[T_{i,j}]
\]

\[
\leq \frac{1}{2} \sum_{i,j} M_{ij} |x_i - x_j|(|x_i| + |x_j|)
\]

\[
= \frac{1}{2} a \cdot b
\]

\[
\leq \frac{1}{2} \|a\| \|b\|
\]
Recall $\delta = \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$, $a_{ij} = \sqrt{M_{ij}} |x_i - x_j|$, $b_{ij} = \sqrt{M_{ij}} |x_i| + |x_j|$
Recall \( \delta = \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2} \), \( a_{ij} = \sqrt{M_{ij}} |x_i - x_j| \), \( b_{ij} = \sqrt{M_{ij}} |x_i| + |x_j| \)

\[
\|a\|^2 = \sum_{i,j} M_{ij} (x_i - x_j)^2 = \frac{\delta}{n} \sum_{i,j} (x_i - x_j)^2
\]

\[
= \frac{\delta}{n} \sum_{i,j} (x_i^2 + x_j^2) - \sum_{i,j} 2x_i x_j
\]

\[
= \frac{\delta}{n} \sum_{i,j} (x_i^2 + x_j^2) - 2(\sum_{i} x_i)^2
\]

\[
\leq \frac{\delta}{n} \sum_{i,j} (x_i^2 + x_j^2) = 2\delta \sum_{i} x_i^2
\]
Recall $\delta = \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$, $a_{ij} = \sqrt{M_{ij}} |x_i - x_j|$, $b_{ij} = \sqrt{M_{ij}} |x_i| + |x_j|$

\[
\|a\|^2 = \sum_{i,j} M_{ij} (x_i - x_j)^2 = \frac{\delta}{n} \sum_{i,j} (x_i - x_j)^2 \\
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\leq \frac{\delta}{n} \sum_{i,j} (x_i^2 + x_j^2) = 2\delta \sum_i x_i^2
\]

\[
\|b\|^2 = \sum_{i,j} M_{ij} (|x_i| + |x_j|)^2 \\
\leq \sum_{i,j} M_{ij} (2x_i^2 + 2x_j^2) \\
= 4 \sum_i x_i^2
\]
Goal: \[ \mathbb{E}_{S \sim D} \left[ \left\{ \frac{1}{d} \left| E(S, V-S) \right| \right\} \right] \leq \sqrt{2\delta} \]
Goal: \[ \frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S \sim D}[\min(|S|,|V-S|)]} \leq \sqrt{2\delta} \]

Numerator:
\[ \mathbb{E}_{S \sim D}[\frac{1}{d}|E(S,V-S)|] \leq \frac{1}{2} \| a \| \| b \| \]
\[ \leq \frac{1}{2} \sqrt{2\delta} \sum_i x_i^2 \sqrt{4 \sum_i x_i^2} = \sqrt{2\delta} \sum_i x_i^2 \]

Recall Denominator:
\[ \mathbb{E}_{S \sim D}[\min(|S|,|V-S|)] = \sum_i x_i^2 \]

We get
\[ \frac{\mathbb{E}_{S \sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S \sim D}[\min(|S|,|V-S|)]} \leq \sqrt{2\delta} \]
Goal: \[ \frac{\mathbb{E}_{S \sim D} \left[ \frac{1}{d} |E(S, V - S)| \right]}{\mathbb{E}_{S \sim D} \left[ \min(|S|, |V - S|) \right]} \leq \sqrt{2\delta} \]

Numerator:
\[ \mathbb{E}_{S \sim D} \left[ \frac{1}{d} |E(S, V - S)| \right] \leq \frac{1}{2} \|a\| \|b\| \]
\[ \leq \frac{1}{2} \sqrt{2\delta \sum_{i} x_i^2} \sqrt{4 \sum_{i} x_i^2} = \sqrt{2\delta \sum_{i} x_i^2} \]

Recall Denominator:
\[ \mathbb{E}_{S \sim D} \left[ \min(|S|, |V - S|) \right] = \sum_{i} x_i^2 \]

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\[ \frac{\mathbb{E}_{S \sim D} \left[ \frac{1}{d} |E(S, V - S)| \right]}{\mathbb{E}_{S \sim D} \left[ \min(|S|, |V - S|) \right]} \leq \sqrt{2\delta} \]

Thus \( \exists S_i \) such that \( h(S_i) \leq \sqrt{2\delta}, \) which gives \( h(G) \leq \sqrt{2(1 - \lambda)} \) \( \square \)
Cycle

Tight example for hard part of Cheeger?
Tight example for hard part of Cheeger?

$$\mu \geq 1 - \lambda^2 \leq h(G) = \sqrt{2} \left(1 - \lambda^2\right) = \sqrt{2} \mu$$

Will show other side of Cheeger is asymptotically tight.

Cycle on $n$ nodes.

Edge expansion: Cut in half.

$$|S| = \frac{n}{2}, \quad |E(S, S)| \rightarrow h(G) = 4\frac{n}{2}.$$
Cycle

Tight example for hard part of Cheeger?

\[ \frac{\mu}{2} = \frac{1 - \lambda_2}{2} \]
Tight example for hard part of Cheeger?

\[ \frac{\mu}{2} = \frac{1 - \lambda^2}{2} \leq h(G) \]
Tight example for hard part of Cheeger?

\[ \frac{\mu}{2} = \frac{1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)} \]
Tight example for hard part of Cheeger?

\[ \frac{\mu}{2} = \frac{1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)} = \sqrt{2\mu} \]
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Will show other side of Cheeger is asymptotically tight.

Cycle on \( n \) nodes.

Edge expansion: Cut in half.

\[ |S| = \frac{n}{2}, |E(S, \overline{S})| = 2 \]
Tight example for hard part of Cheeger?
\[
\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}
\]
Will show other side of Cheeger is asymptotically tight.

Cycle on \( n \) nodes.

Edge expansion: Cut in half.
\[
|S| = \frac{n}{2}, \quad |E(S, S^c)| = 2
\]
\[
\rightarrow h(G) = \frac{4}{n}.
\]
Tight example for hard part of Cheeger?

\[ \frac{\mu}{2} = \frac{1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)} = \sqrt{2\mu} \]

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Edge expansion: Cut in half.

\[ |S| = \frac{n}{2}, \ |E(S, \overline{S})| = 2 \]

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Show eigenvalue gap \( \mu \) is \( O\left( \frac{1}{n^2} \right) \).
Tight example for hard part of Cheeger?

\[ \frac{\mu}{2} = \frac{1 - \lambda_2}{2} \leq h(G) \leq \sqrt{2(1 - \lambda_2)} = \sqrt{2\mu} \]

Will show other side of Cheeger is asymptotically tight.

Cycle on \( n \) nodes.

Edge expansion: Cut in half.

- \( |S| = \frac{n}{2} \), \( |E(S, \overline{S})| = 2 \)
- \( \Rightarrow h(G) = \frac{4}{n} \).

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Find \( x \perp 1 \) with Rayleigh quotient, \( \frac{x^T Mx}{x^T x} \) close to 1.
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$$x_i = \begin{cases} i - n/4 & \text{if } i \leq n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$$

Asymptotically tight example for upper bound for Cheeger $h(G) \leq \sqrt{2} (1 - \lambda_2)$ = $\sqrt{2} \mu$. 

$$\mu = \lambda_1 - \lambda_2 = O\left(\frac{1}{n^2}\right)$$
Find $x \perp 1$ with Rayleigh quotient, \( \frac{x^T M x}{x^T x} \) close to 1.

\[
X_i = \begin{cases} 
  i - n/4 & \text{if } i \leq n/2 \\
  3n/4 - i & \text{if } i > n/2
\end{cases}
\]

Hit with $M$.

\[
(Mx)_i = \begin{cases} 
  -n/4 + 1/2 & \text{if } i = 1, n \\
  n/4 - 1 & \text{if } i = n/2 \\
  x_i & \text{otherwise}
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\[
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$$\rightarrow x^T M x = x^T x(1 - O(\frac{1}{n^2})) \rightarrow \lambda_2 \geq 1 - O(\frac{1}{n^2})$$
Find \( x \perp 1 \) with Rayleigh quotient, \( \frac{x^T M x}{x^T x} \) close to 1.

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\]

\[
\to x^T M x = x^T x (1 - O\left(\frac{1}{n^2}\right)) \to \lambda_2 \geq 1 - O\left(\frac{1}{n^2}\right)
\]

\[
\mu = \lambda_1 - \lambda_2 = O\left(\frac{1}{n^2}\right)
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$$(Mx)_i = \begin{cases} 
  -n/4 + 1/2 & \text{if } i = 1, n \\
  n/4 - 1 & \text{if } i = n/2 \\
  x_i & \text{otherwise}
\end{cases}$$

$$x^T M x = x^T x(1 - O(\frac{1}{n^2})) \quad \Rightarrow \quad \lambda_2 \geq 1 - O(\frac{1}{n^2})$$

$$\mu = \lambda_1 - \lambda_2 = O(\frac{1}{n^2})$$

$$h(G) = \frac{4}{n} = \Theta(\sqrt{2\mu})$$
Find $x \perp 1$ with Rayleigh quotient, $\frac{x^T M x}{x^T x}$ close to 1.

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n/4 - 1 & \text{if } i = n/2 \\
\lambda_i & \text{otherwise}
\end{cases} \\
\end{align*}$$

$$\rightarrow x^T M x = x^T x (1 - O(\frac{1}{n^2})) \rightarrow \lambda_2 \geq 1 - O(\frac{1}{n^2})$$

$$\mu = \lambda_1 - \lambda_2 = O(\frac{1}{n^2})$$

$$h(G) = \frac{4}{n} = \Theta(\sqrt{2\mu})$$

Asymptotically tight example for upper bound for Cheeger

$$h(G) \leq \sqrt{2(1 - \lambda_2)} = \sqrt{2\mu}.$$
Sum up.

$1 - \lambda_2$ as a relaxation of $\phi(G)$. 

Sum up.

$1 - \lambda_2$ as a relaxation of $\phi(G)$.

Sweeping cut Algorithm
Sum up.

$1 - \lambda_2$ as a relaxation of $\phi(G)$.

Sweeping cut Algorithm

Probabilistic argument to show there exists a good threshold cut
$1 - \lambda_2$ as a relaxation of $\phi(G)$.
Sweeping cut Algorithm
Probabilistic argument to show there exists a good threshold cut
Example: Cycle, Cheeger hard part is asymptotic tight.
Satish will be back on Tuesday.
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