Welcome back.

Today.

Review: Spectral gap, Edge expansion $h(G)$, Sparsity $\phi(G)$ etc.

Write $1-\lambda_2$ as a relaxation of $\phi(G)$, Cheeger easy part

Cheeger hard part: Sweeping cut Algorithm, Proof, Asymptotic tight example

Edge Expansion/Conductance.

Graph $G=(V,E)$,

Assume regular graph of degree d.

Edge Expansion.

$h(S)=\frac{|E(S,V-S)|}{d\min(|S|,|V-S|)}$, $h(G)=\min_{S\subset V} h(S)$

Conductance (Sparsity).

$\phi(S)=\frac{n|E(S,V-S)|}{d|S||V-S|}$, $\phi(G)=\min_{S\subset V}\phi(S)$

Note $n\geq \max(|S|,|V|-|S|)\geq n/2$

$\rightarrow h(G)\leq\phi(G)\leq 2h(S)$

Spectra of the graph.

A: Adjacency Matrix $A_{ij}=1 \Leftrightarrow (i,j)\in E$

$M=\frac{1}{d}A$, normalized adjacency matrix, M real, symmetric

orthonormal eigenvectors: $v_1,\ldots,v_n$ with eigenvalues $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$

Claim: Any two eigenvectors with different eigenvalues are orthogonal.

Proof: Eigenvectors: v,v' with eigenvalues λ,λ'.

$v^TMv'=v^T(\lambda' v')=\lambda' v^Tv'$

$v^TMv'=\lambda v^Tv'=\lambda v^Tv$. □

Action of M.

v - assigns values to vertices.

$(Mv)_i=\frac{1}{d}\sum_{j\sim i} v_j$.

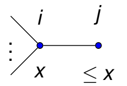
Action of M: taking the average of your neighbours.

(Direct) result from the action of M, $|\lambda_i|\leq 1 \quad \forall i$

$v_1=\mathbf{1}$. $\lambda_1=1$.

Claim: For a connected graph $\lambda_2<1$.

Proof: Second Eigenvector: $v\perp\mathbf{1}$. Max value x.
Connected $\rightarrow$ path from x valued node to lower value.
$\rightarrow \exists\, e=(i,j)$, $v_i=x$, $x_j<x$.



$(Mv)_i\leq\frac{1}{d}(x+x\cdots+v_j)<x$.

Therefore $\lambda_2<1$. □

Claim: Connected if $\lambda_2<1$.

Proof: By contradiction. Assign $+1$ to vertices in one component, $-\delta$ to rest.

$x_i=(Mx_i) \implies$ eigenvector with $\lambda=1$.

Choose δ to make $\sum_i x_i=0$, i.e., $x\perp\mathbf{1}$. □

Spectral Gap and the connectivity of graph.

Spectral gap: $\mu=\lambda_1-\lambda_2=1-\lambda_2$.

Recall: $h(G)=\min_{S,|S|\leq|V|/2}\frac{|E(S,V-S)|}{|S|}$

$1-\lambda_2=0\Leftrightarrow\lambda_2=1\Leftrightarrow G$ disconnected $\Leftrightarrow h(G)=0$

In general, small spectral gap $1-\lambda_2$ suggests "poorly connected" graph

Formally

Cheeger's Inequality

$$\frac{1-\lambda_2}{2}\leq h(G)\leq\sqrt{2(1-\lambda_2)}$$

Spectral Gap and Conductance.

We will show $1-\lambda_2$ as a continuous relaxation of $\phi(G)$.

$$\phi(G)=min_{S\in V}\frac{n|E(S,V-S)|}{d|S||V-S|}$$

Let x be the characteritic vector of set S $\quad x_i=\begin{cases}1 & \text{if } i\in S\\ 0 & \text{if } i\notin S\end{cases}$

$$|E(S,V-S)|=\frac{1}{2}\sum_{i,j}A_{ij}|x_i-x_j|=\frac{d}{2}\sum_{i,j}M_{ij}(x_i-x_j)^2$$

$$|S||V-S|=\frac{1}{2}\sum_{i,j}|x_i-x_j|=\frac{1}{2}\sum_{i,j}(x_i-x_j)^2$$

$$\phi(G)=min_{x\in\{0,1\}^V-\{\mathbf{0},\mathbf{1}\}}\frac{n\sum_{i,j}M_{ij}(x_i-x_j)^2}{\sum_{i,j}(x_i-x_j)^2}$$

Recall Rayleigh Quotient: $\lambda_2 = max_{x \in \mathbb{R}^V - \{\mathbf{0}\}, x \perp \mathbf{1}} \frac{x^T Mx}{x^T x}$

$$1 - \lambda_2 = min_{x \in \mathbb{R}^V - \{\mathbf{0}\}, x \perp \mathbf{1}} \frac{2(x^T x - x^T Mx)}{2x^T x}$$

Claim: $2x^T x = \frac{1}{n} \sum_{i,j} (x_i - x_j)^2$

Proof:

$$\sum_{i,j} (x_i - x_j)^2 = \sum_{i,j} x_i^2 + x_j^2 - 2x_i x_j$$
$$= 2n \sum_i x_i^2 - 2(\sum_i x_i)^2 = 2n \sum_i x_i^2 = 2nx^T x$$

We used $x \perp \mathbf{1} \Rightarrow \sum_i x_i = 0$ □

Recall Rayleigh Quotient: $\lambda_2 = max_{x \in \mathbb{R}^V - \{\mathbf{0}\}, x \perp \mathbf{1}} \frac{x^T Mx}{x^T x}$

$$1 - \lambda_2 = min_{x \in \mathbb{R}^V - \{\mathbf{0}\}, x \perp \mathbf{1}} \frac{2(x^T x - x^T Mx)}{2x^T x}$$

Claim: $2(x^T x - x^T Mx) = \sum_{i,j} M_{ij} (x_i - x_j)^2$

Proof:

$$\sum_{i,j} M_{ij} (x_i - x_j)^2 = \sum_{i,j} M_{ij} (x_i^2 + x_j^2) - 2 \sum_{i,j} M_{ij} x_i x_j$$
$$= \sum_i \sum_{j \sim i} \frac{1}{d} (x_i^2 + x_j^2) - 2x^T Mx$$
$$= 2 \sum_{(i,j) \in E} \frac{1}{d} (x_i^2 + x_j^2) - 2x^T Mx$$
$$= 2 \sum_i x_i^2 - 2x^T Mx = 2x^T x - 2x^T Mx$$

□

Combining the two claims, we get

$$1 - \lambda_2 = min_{x \in \mathbb{R}^V - \{\mathbf{0}\}, x \perp \mathbf{1}} \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$$
$$= min_{x \in \mathbb{R}^V - \text{Span}\{\mathbf{1}\}} \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$$

Recall

$$\phi(G) = min_{x \in \{0,1\}^V - \{\mathbf{0}, \mathbf{1}\}} \frac{n \sum_{i,j} M_{ij} (x_i - x_j)^2}{\sum_{i,j} (x_i - x_j)^2}$$

We have $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$, thus

$$1 - \lambda_2 \le \phi(G) \le 2h(G)$$

Hooray!! We get the easy part of Cheeger $\frac{1 - \lambda_2}{2} \le h(G)$

Cheeger Hard Part.

Now let's get to the hard part of Cheeger $h(G) \le \sqrt{2(1 - \lambda_2)}$.

Idea: We have $1 - \lambda_2$ as a continuous relaxation of $\phi(G)$

Take the 2^{nd} eigenvector $x = argmin_{x \in \mathbb{R}^V - \text{Span}\{\mathbf{1}\}} \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$

Consider x as an embedding of the vertices to the real line.

Round x to get a $x \in \{0,1\}^V$

Rounding: Take a threshold t,

$$\begin{cases} x_i \ge t & \rightarrow x_i = 1 \\ x_i < t & \rightarrow x_i = 0 \end{cases}$$

What will be a good t?

We don't know. Try all possible thresholds ($n - 1$ possibilities), and hope there is a t leading to a good cut!

Sweeping Cut Algorithm

Input: $G = (V, E)$, $x \in \mathbb{R}^V, x \perp \mathbf{1}$

Sort the vertices in non-decreasing order in terms of their values in x

WLOG $V = \{1, \ldots, n\}$ $\quad x_1 \le x_2 \le \ldots \le x_n$

Let $S_i = \{1, \ldots, i\}$ $\quad i = 1, \ldots, n - 1$

Return $S = argmin_{S_i} h(S_i)$

Main Lemma: $G = (V, E)$, d-regular

$x \in \mathbb{R}^V, x \perp \mathbf{1}, \delta = \frac{\sum_{i,j} M_{ij} (x_i - x_j)^2}{\frac{1}{n} \sum_{i,j} (x_i - x_j)^2}$

If S is the ouput of the sweeping cut algorithm, then $h(S) \le \sqrt{2\delta}$

Note: Applying the Main Lemma with the 2^{nd} eigenvector v_2, we have $\delta = 1 - \lambda_2$, and $h(G) \le h(S) \le \sqrt{2(1 - \lambda_2)}$. Done!

Proof of Main Lemma

WLOG $V = \{1, \ldots, n\}$ $\quad x_1 \le x_2 \le \ldots \le x_n$

Want to show

$$\exists i \text{ s.t. } h(S_i) = \frac{\frac{1}{d} |E(S, V - S)|}{min(|S|, |V - S|)} \le \sqrt{2\delta}$$

Probabilistic Argument: Construct a distribution D over $\{S_1, \ldots, S_{n-1}\}$ such that

$$\frac{\mathbb{E}_{S \sim D} [\frac{1}{d} |E(S, V - S)|]}{\mathbb{E}_{S \sim D} [min(|S|, |V - S|)]} \le \sqrt{2\delta}$$

$$\rightarrow \mathbb{E}_{S \sim D} [\frac{1}{d} |E(S, V - S)| - \sqrt{2\delta} min(|S|, |V - S|)] \le 0$$

$$\exists S \quad \frac{1}{d} |E(S, V - S)| - \sqrt{2\delta} min(|S|, |V - S|) \le 0$$

The distribution D

WLOG, shift and scale so that $x_{\lfloor \frac{n}{2} \rfloor} = 0$, and $x_1^2 + x_n^2 = 1$

Take t from the range $[x_1, x_n]$ with density function $f(t) = 2|t|$.

Check: $\int_{x_1}^{x_n} f(t)\mathrm{d}t = \int_{x_1}^{0} -2t\mathrm{d}t + \int_{0}^{x_n} 2t\mathrm{d}t = x_1^2 + x_n^2 = 1$

$S = \{i : x_i \le t\}$

Take D as the distribution over $S_1, \dots, S_{n-1}$ resulted from the above procedure.

Cauchy-Schwarz Inequality

$|a \cdot b| \le \|a\|\|b\|$, as $a \cdot b = \|a\|\|b\| cos(a, b)$

Applying with $a, b \in \mathbb{R}^{n^2}$ with $a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}|x_i| + |x_j|$

Numerator:

$$\begin{aligned}\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V-S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\ &\le \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|) \\ &= \frac{1}{2} a \cdot b \\ &\le \frac{1}{2}\|a\|\|b\|\end{aligned}$$

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \le \sqrt{2\delta}$

Denominator:

Let $T_i = i$ is in the smaller set of $S, V - S$

Can check

$$\mathbb{E}_{S\sim D}[T_i] = Pr[T_i] = x_i^2$$

$$\begin{aligned}\mathbb{E}_{S\sim D}[min(|S|, |V-S|)] &= \mathbb{E}_{S\sim D}[\sum_i T_i] \\ &= \sum_i \mathbb{E}_{S\sim D}[T_i] \\ &= \sum_i x_i^2\end{aligned}$$

Recall $\delta = \frac{\sum_{i,j} M_{ij}(x_i - x_j)^2}{\frac{1}{n}\sum_{i,j}(x_i - x_j)^2}, a_{ij} = \sqrt{M_{ij}}|x_i - x_j|, b_{ij} = \sqrt{M_{ij}}|x_i| + |x_j|$

$$\begin{aligned}\|a\|^2 &= \sum_{i,j} M_{ij}(x_i - x_j)^2 = \frac{\delta}{n}\sum_{i,j}(x_i - x_j)^2 \\ &= \frac{\delta}{n}\sum_{i,j}(x_i^2 + x_j^2) - \sum_{i,j} 2x_i x_j \\ &= \frac{\delta}{n}\sum_{i,j}(x_i^2 + x_j^2) - 2(\sum_i x_i)^2 \\ &\le \frac{\delta}{n}\sum_{i,j}(x_i^2 + x_j^2) = 2\delta \sum_i x_i^2\end{aligned}$$

$$\begin{aligned}\|b\|^2 &= \sum_{i,j} M_{ij}(|x_i| + |x_j|)^2 \\ &\le \sum_{i,j} M_{ij}(2x_i^2 + 2x_j^2) \\ &= 4\sum_i x_i^2\end{aligned}$$

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \le \sqrt{2\delta}$

Numerator:

Let $T_{i,j} = i, j$ is cut by $(S, V - S)$

$$\begin{cases} x_i, x_j \text{ same sign:} & Pr[T_{i,j}] = |x_i^2 - x_j^2| \\ x_i, x_j \text{ different sign:} & Pr[T_{i,j}] = x_i^2 + x_j^2 \end{cases}$$

A common upper bound: $\mathbb{E}[T_{i,j}] = Pr[T_{i,j}] \le |x_i - x_j|(|x_i| + |x_j|)$

$$\begin{aligned}\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V-S)|] &= \frac{1}{2}\sum_{i,j} M_{ij}\mathbb{E}[T_{i,j}] \\ &\le \frac{1}{2}\sum_{i,j} M_{ij}|x_i - x_j|(|x_i| + |x_j|)\end{aligned}$$

Goal: $\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S,V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|,|V-S|)]} \le \sqrt{2\delta}$

Numerator:

$$\begin{aligned}\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V-S)|] &=\le \frac{1}{2}\|a\|\|b\| \\ &\le \frac{1}{2}\sqrt{2\delta\sum_i x_i^2}\sqrt{4\sum_i x_i^2} \quad = \sqrt{2\delta}\sum_i x_i^2\end{aligned}$$

Recall **Denominator:**

$$\mathbb{E}_{S\sim D}[min(|S|, |V-S|)] = \sum_i x_i^2$$

We get

$$\frac{\mathbb{E}_{S\sim D}[\frac{1}{d}|E(S, V-S)|]}{\mathbb{E}_{S\sim D}[min(|S|, |V-S|)]} \le \sqrt{2\delta}$$

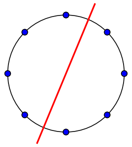
Thus $\exists S_i$ such that $h(S_i) \le \sqrt{2\delta}$, which gives $h(G) \le \sqrt{2(1-\lambda)}$ □

Cycle

Tight example for hard part of Cheeger?

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Will show other side of Cheeger is asymptotically tight.



Cycle on n nodes.

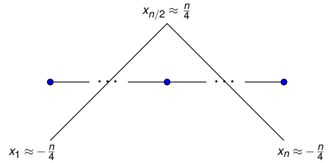
Edge expansion:Cut in half.
$|S| = \frac{n}{2}$, $|E(S,\overline{S})| = 2$
$\rightarrow h(G) = \frac{4}{n}$.

Show eigenvalue gap μ is $O(\frac{1}{n^2})$.

Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^T M x}{x^T x}$ close to 1.

Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^T M x}{x^T x}$ close to 1.

$$x_i = \begin{cases} i - n/4 & \text{if } i \leq n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$$



Hit with M.

$$(Mx)_i = \begin{cases} -n/4 + 1/2 & \text{if } i = 1, n \\ n/4 - 1 & \text{if } i = n/2 \\ x_i & \text{otherwise} \end{cases}$$

$\rightarrow x^T M x = x^T x(1 - O(\frac{1}{n^2})) \quad \rightarrow \lambda_2 \geq 1 - O(\frac{1}{n^2})$

$\mu = \lambda_1 - \lambda_2 = O(\frac{1}{n^2})$

$h(G) = \frac{4}{n} = \Theta(\sqrt{2\mu})$

Asymptotically tight example for upper bound for Cheeger
$h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$.

Sum up.

$1 - \lambda_2$ as a relaxation of $\phi(G)$.
Sweeping cut Algorithm
Probabilistic argument to show there exists a good threshold cut
Example: Cycle, Cheeger hard part is asymptotic tight .

Satish will be back on Tuesday.