Welcome back.

Today.

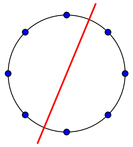
Continue Sampling combinatorial structures.
Random Walks.
Spectral Gap/Mixing Time.
Expansion/Spectral Gap: Cheeger
Example: partial orders.
Cheeger and Tight Examples.

Cycle

Tight example for Other side of Cheeger?

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Will show other side of Cheeger is tight.



Cycle on n nodes.

Edge expansion:Cut in half.
$|S| = \frac{n}{2}$, $|E(S,\overline{S})| = 2$
$\rightarrow h(G) = \frac{4}{n}$.

Show eigenvalue gap $\mu \leq \frac{1}{n^2}$.

Find $x \perp \mathbf{1}$ with Rayleigh quotient, $\frac{x^T M x}{x^T x}$ close to 1.

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$$x_i = \begin{cases} i - n/4 & \text{if } i \leq n/2 \\ 3n/4 - i & \text{if } i > n/2 \end{cases}$$

Hit with M.

$$(Mx)_i = \begin{cases} -n/4 + 1/2 & \text{if } i = 1, n \\ n/4 - 1 & \text{if } i = n/2 \\ x_i & \text{otherwise} \end{cases}$$

$\rightarrow x^T M x = x^T x (1 - O(\frac{1}{n^2}))$ $\quad \rightarrow \lambda_2 \geq 1 - O(\frac{1}{n^2})$

$\mu = \lambda_1 - \lambda_2 = O(\frac{1}{n^2})$

$h(G) = \frac{2}{n} = \Theta(\sqrt{\mu})$

$\frac{\mu}{2} = \frac{1-\lambda_2}{2} \leq h(G) \leq \sqrt{2(1-\lambda_2)} = \sqrt{2\mu}$

Tight example for upper bound for Cheeger.

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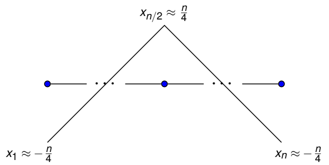

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Tight example for upper bound for Cheeger.

Eigenvalues of cycle?

Eigenvalues: $\cos \frac{2\pi k}{n}$.

$x_i = \cos \frac{2\pi k i}{n}$

$(Mx)_i = \cos\left(\frac{2\pi k(i+1)}{n}\right) + \cos\left(\frac{2\pi k(i-1)}{n}\right) = 2\cos\left(\frac{2\pi k}{n}\right)\cos\left(\frac{2\pi k i}{n}\right)$

Eigenvalue: $\cos \frac{2\pi k}{n}$.

Eigenvalues:
vibration modes of system.
Fourier basis.

Sum up.

Sampling by random walks.
Random Walks mix if μ is "large".
If expanding μ is large. "Cheeger.
Example: partial orders.
More songs about People and Food ...or Cheeger.

See you on Tuesday. Not!