CS270: Lecture 2.

Admin:

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Admin:
Check Piazza.

CS270: Lecture 2.

Admin:
Check Piazza. There is a poll on bspace.

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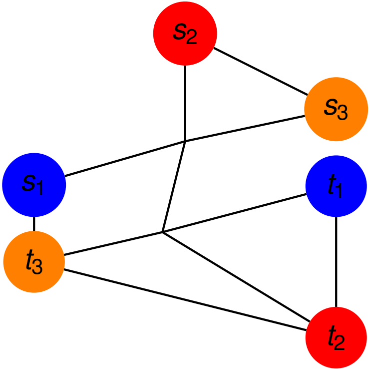
Today:

- Finish Path Routing.
- Games

Path Routing.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of k paths connecting s_i and t_i and minimize max load on any edge.



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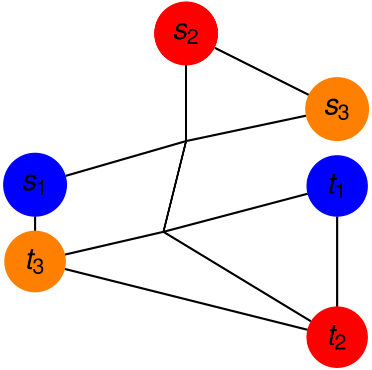

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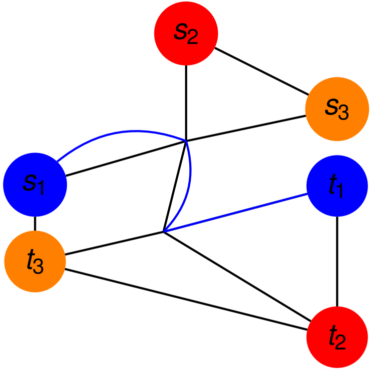

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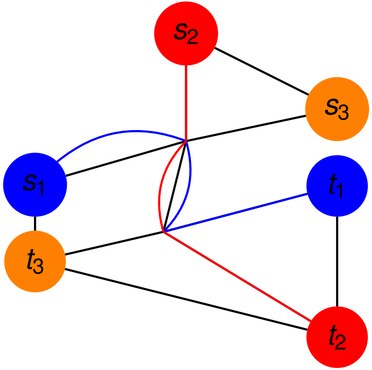

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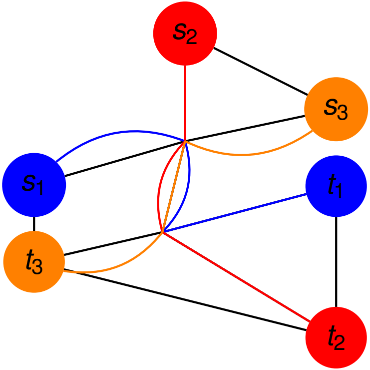

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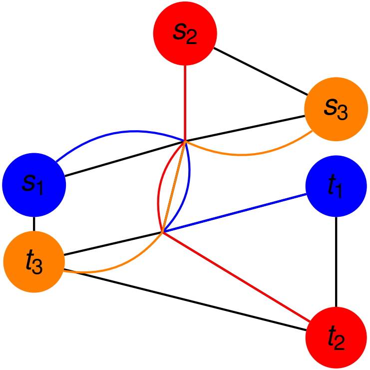

Value: 3

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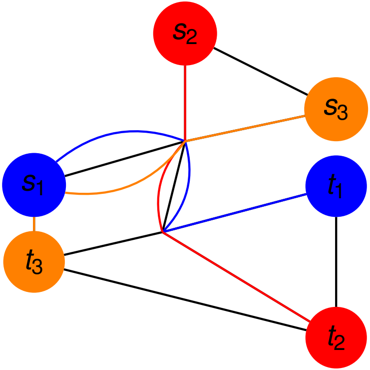

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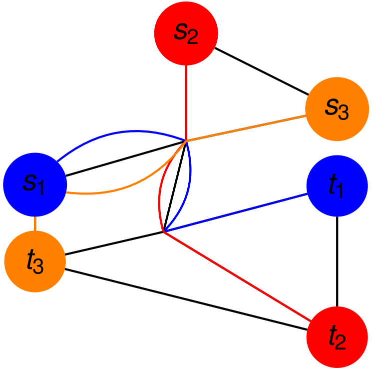

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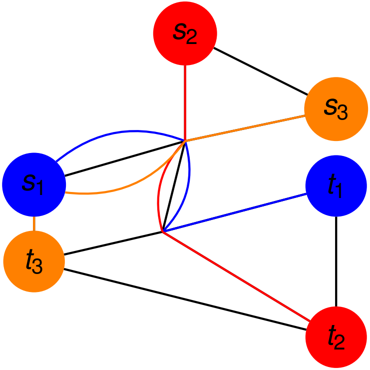

~~Value: 3~~

Value: 2

Toll problem.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of k paths assign one unit of "toll" to edges to maximize total toll for connecting pairs.

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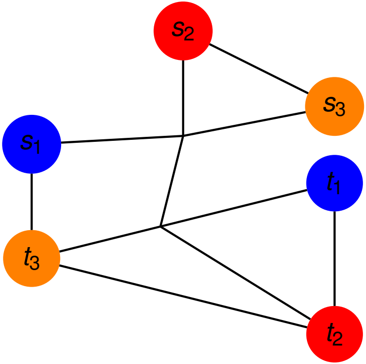

Assign $\frac{1}{11}$ on each of 11 edges.

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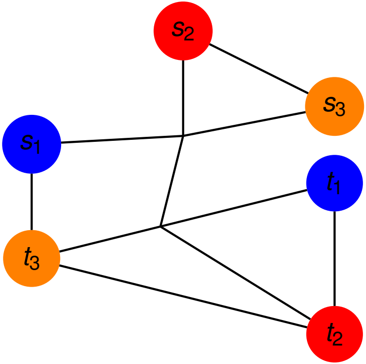

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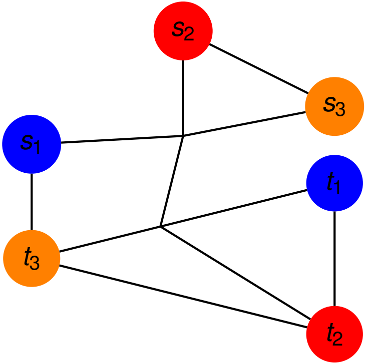

Assign $\frac{1}{11}$ on each of 11 edges.
Total toll: $\frac{3}{11}+\frac{3}{11}+\frac{3}{11}$

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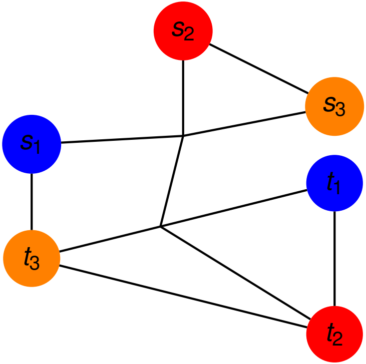

Assign $\frac{1}{11}$ on each of 11 edges.
Total toll: $\frac{3}{11} + \frac{3}{11} + \frac{3}{11} = \frac{9}{11}$

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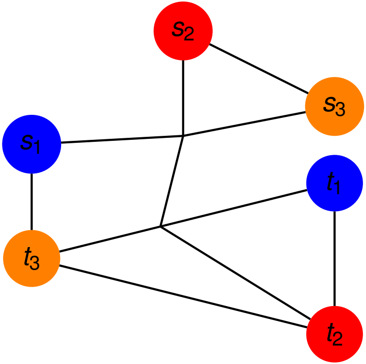

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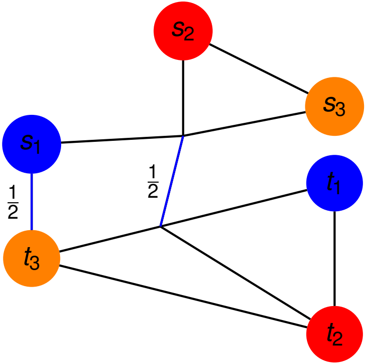

Assign $\frac{1}{11}$ on each of 11 edges.

Total toll: $\frac{3}{11} + \frac{3}{11} + \frac{3}{11} = \frac{9}{11}$

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Assign $1/2$ on these two edges.

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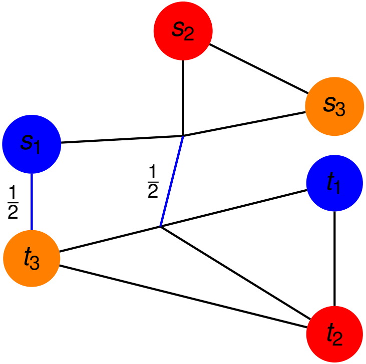

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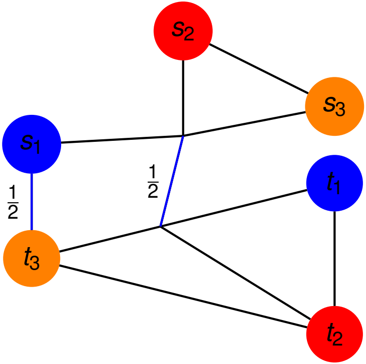

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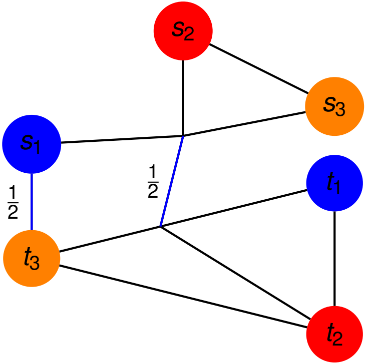

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Toll is lower bound on Path Routing.

From before:
Max bigger than minimum weighted average:
$\max_e c(e) \geq \sum_e c(e)d(e)$
Total length is total congestion: $\sum_e c(e)d(e) = \sum_i d(p_i)$

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A toll solution is lower bound on any routing solution.

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A toll solution is lower bound on any routing solution.
Any routing solution is an upper bound on a toll solution.

Algorithm.

Assign tolls.

Algorithm.

Assign tolls.
How to route?

Algorithm.

Assign tolls.

How to route? Shortest paths!

Algorithm.

Assign tolls.
How to route? Shortest paths!
Assign routing.

Algorithm.

Assign tolls.
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How to assign tolls?

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Assign tolls.
How to route? Shortest paths!
Assign routing.
How to assign tolls? Higher tolls on congested edges.

Algorithm.

Assign tolls.
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Toll: $d(e) \propto 2^{c(e)}$.

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Equilibrium:

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Equilibrium:
The shortest path routing has <u>has</u> $d(e) \propto 2^{c(e)}$.

Algorithm.

Assign tolls.
How to route? Shortest paths!
Assign routing.
How to assign tolls? Higher tolls on congested edges.
Toll: $d(e) \propto 2^{c(e)}$.

Equilibrium:
The shortest path routing has has $d(e) \propto 2^{c(e)}$.

The routing does not change, the tolls do not change.

How good is equilibrium?

Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.

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Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.

$$c_{opt} \geq \sum_i d(s_i, t_i) = \sum_e d(e)c(e)$$

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\begin{aligned}
c_{opt} &\geq \sum_i d(s_i, t_i) = \sum_e d(e) c(e) \\
&= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e)
\end{aligned}
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&\geq \frac{\sum_{e:c(e)>c_t} 2^{c(e)} c(e)}{\sum_{e:c(e)>c_t} 2^{c(e)} + \sum_{e:c(e)\leq c_t} 2^{c(e)}}
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For e with $c(e) \le c_{max} - 2\log m$; $2^{c(e)} \le 2^{c_{\max} - 2\log m} = \frac{2^{c_{max}}}{m^2}$.

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Let $c_t = c_{max} - 2\log m$.

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Or $c_{max} \leq (1+\frac{1}{m}) c_{opt} + 2\log m$.

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Or $c_{max} \leq (1 + \frac{1}{m}) c_{opt} + 2\log m$.
(Almost) within $2\log m$ of optimal!

The end: sort of.

Got to here in class. Feel free to continue reading.

Getting to equilibrium.

Maybe no equilibrium!

Getting to equilibrium.

Maybe no equilibrium!

Approximate equilibrium:

Getting to equilibrium.

Maybe no equilibrium!

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Each path is routed along a path with length
within a factor of 3 of the shortest path and $d(e) \propto 2^{c(e)}$.

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Lose a factor of three at the beginning.
$c_{opt} \geq \sum_i d(s_i, t_i) \geq \frac{1}{3} \sum_e d(p_i)$.

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We obtain $c_{max} = 3(1 + \frac{1}{m})c_{opt} + 2 \log m$.

Getting to equilibrium.

Maybe no equilibrium!

Approximate equilibrium:

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This is worse!
What do we gain?

An algorithm!

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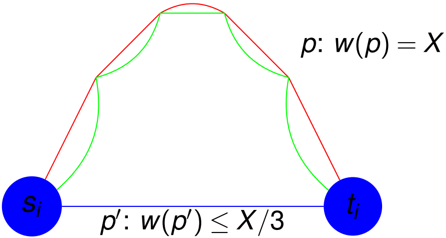

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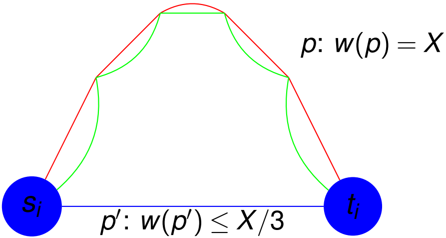

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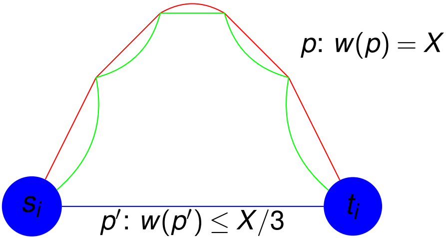

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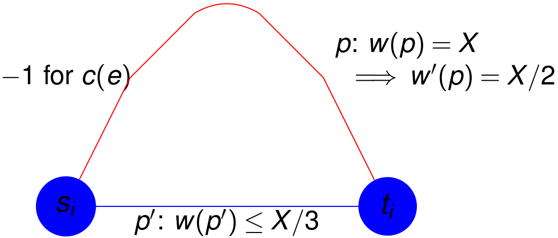

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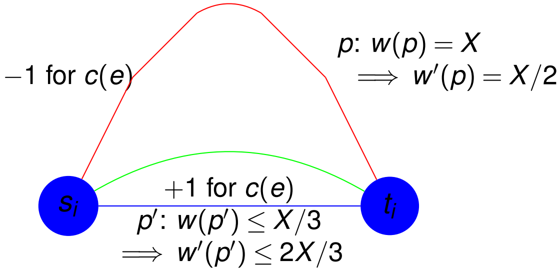

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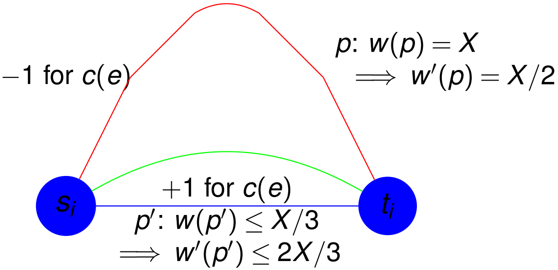

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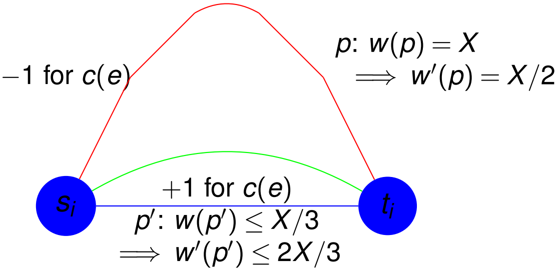

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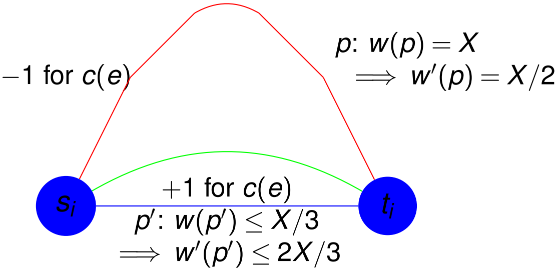

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Potential function decreases. $\implies$ termination and existence.

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Fractional paths?

Wrap up.

Dueling players:

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A lower bound is “necessary” (natural),
and helpful (mysterious?)!

Strategic Games.

N players.

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Each player has strategy set. $\{S_1, \ldots, S_N\}$.

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	C	**D**
C	(3,3)	(0,5)
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Nash Equilibrium: neither player has incentive to change strategy.

Digression..

What situations?

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This class(today): simpler version.

Two Person Zero Sum Games

2 players.

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Roshambo: rock,paper, scissors.

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R	0	1	-1
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Mixed Strategies.

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How do you play?

Mixed Strategies.

		R	P	S
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

How do you play?

Player 1: play each strategy with equal probability.

Mixed Strategies.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
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Player 2: play each strategy with equal probability.

Mixed Strategies.

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Mixed strategies: Each player plays distribution over strategies.

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Definitions.

Mixed strategies: Each player plays distribution over strategies.

Pure strategies: Each player plays single strategy.

Payoffs: Equilibrium.

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Payoffs?

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

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S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff.

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
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Each player chooses independently:

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
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Payoffs? Can't just look it up in matrix!.

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Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].$$

Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = 0.$$

[1] Remember zero sum games have one payoff.

Payoffs: Equilibrium.

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Payoffs? Can't just look it up in matrix!.

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Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$
Random variable X (payoff).

$$E[X] = \sum_{(i,j)} X(i,j)Pr[(i,j)].$$

Each player chooses independently:
$Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

$$E[X] = 0.^{1}$$

[1] Remember zero sum games have one payoff.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$

Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
Expected payoff of Scissors?

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
Expected payoff of Scissors? $\frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0.$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0.$
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No better pure strategy.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
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Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
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Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0.$
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No better pure strategy. $\Longrightarrow$ No better mixed strategy!

Mixed strat. payoff is weighted av. of payoffs of pure strats.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j)$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)}(Pr[i] \times Pr[j])X(i,j) = \sum_i Pr[i](\sum_j Pr[j] \times X(i,j))$

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

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Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change!

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

Mixed strategy can't be better than the best pure strategy.

Player 1 has no incentive to change! Same for player 2.

Equilibrium

		R	P	S
		$.3\overline{3}$	$.3\overline{3}$	$.3\overline{3}$
R	$.3\overline{3}$	0	1	-1
P	$.3\overline{3}$	-1	0	1
S	$.3\overline{3}$	1	-1	0

Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? $\frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$.
Expected payoff of Paper? $\frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0$.
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Mixed strat. payoff is weighted av. of payoffs of pure strats.

$E[X] = \sum_{(i,j)} (Pr[i] \times Pr[j]) X(i,j) = \sum_i Pr[i] (\sum_j Pr[j] \times X(i,j))$

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Equilibrium!

Another example plus notation.

Rock, Paper, Scissors, prEempt.

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Payoffs.

Another example plus notation.

Rock, Paper, Scissors, prEempt.

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Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium?

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**.

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.

Notation:

Another example plus notation.

Rock, Paper, Scissors, prEempt.

PreEmpt ties preEmpt, beats everything else.

Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.

Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.

Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Payoffs.

	R	P	S	E
R	0	1	-1	1
P	-1	0	1	1
S	1	-1	0	1
E	-1	-1	-1	0

Equilibrium? **(E,E)**. Pure strategy equilibrium.
Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

Playing the boss...

Row has extra strategy:Cheat.

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Ties with rock and scissors, beats paper. (Scissors, or no rock!)

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Payoff matrix:

Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

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Note: column knows row cheats.

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Why play?
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Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note: column knows row cheats.
Why play?
Row is column's advisor.
... boss.

Equilibrium: play the boss...

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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Payoff? Remember: weighted average of pure strategies.

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Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1$

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Both only play optimal strategies! Complementary slackness.

Why not play just one? Change payoff for other guy!

Lecture 2 ended here..and Lecture 3 reviewed a few of the previous slides and continued into lecture 3 notes.

Two person zero sum games.

$m \times n$ payoff matrix A.

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Row mixed strategy: $x = (x_1, \ldots, x_m)$.

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Payoff for strategy pair (x, y):

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Payoff for strategy pair (x, y):

$$p(x, y) = x^t A y$$

That is,

$$\sum_i x_i \left(\sum_j a_{i,j} y_j \right) = \sum_j \left(\sum_i x_i a_{i,j} \right) y_j.$$

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$$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^*.$$

(No better column strategy, no better row strategy.)

Equilibrium.

Equilibrium pair: (x^*, y^*)?

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[2] $A^{(i)}$ is ith row.

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No row is better:
$\min_i A^{(i)} \cdot y = (x^*)^t A y^*$. [2]

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No row is better:
$\min_i A^{(i)} \cdot y = (x^*)^t A y^*$. [2]

No column is better:
$\max_j (A^t)^{(j)} \cdot x = (x^*)^t A y^*$.

---

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Best Response

Column goes first:

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Find y, where best row is not too low..

$$R = \max_{y} \min_{x} (x^t A y).$$

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Example: Roshambo.

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Example: Roshambo. Value of R?

Best Response

Column goes first:
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Note: x can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of R?

Row goes first:
Find x, where best column is not high.

Best Response

Column goes first:
Find y, where best row is not too low..

$$R = \max_y \min_x (x^t A y).$$

Note: x can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of R?

Row goes first:
Find x, where best column is not high.

$$C = \min_x \max_y (x^t A y).$$

Best Response

Column goes first:
Find y, where best row is not too low..

$$R = \max_y \min_x (x^t A y).$$

Note: x can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of R?

Row goes first:
Find x, where best column is not high.

$$C = \min_x \max_y (x^t A y).$$

Agin: y of form $(0, 0, \ldots, 1, \ldots 0)$.

Best Response

Column goes first:
Find y, where best row is not too low..

$$R = \max_y \min_x (x^t Ay).$$

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Example: Roshambo. Value of R?

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Column goes first:
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Note: x can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshambo. Value of R?

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Duality.

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Weak Duality: $R \leq C$.
Proof: Better to go second. $\square$

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At Equilibrium (x^*, y^*), payoff v:

Duality.

$$R = \max_y \min_x (x^t Ay).$$
$$C = \min_x \max_y (x^t Ay).$$

Weak Duality: $R \leq C$.
Proof: Better to go second. $\square$

At Equilibrium (x^*, y^*), payoff v:
row payoffs (Ay^*) all $\geq v$

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Equilibrium $\implies R = C$!

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Strong Duality: There is an equilibrium point!

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Strong Duality: There is an equilibrium point! and $R = C$!

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Equilibrium $\implies R = C$!

Strong Duality: There is an equilibrium point! and $R = C$!

Doesn't matter who plays first!

Proof of Equilibrium.

Later. Let's see some examples.

An “asymptotic” game.

“Catch me.”

An "asymptotic" game.

"Catch me."

Given: $G = (V, E)$.
Given $a, b \in V$.
Row ("Catch me"): choose path from a to b.
Column("Catcher"): choose edge.
Row pays if column chooses edge on path.

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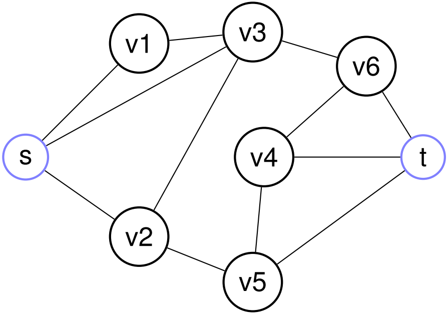
Row pays if column chooses edge on path.

Matrix:
row for each path: p
column for each edge: e
$A[p, e] = 1$ if $e \in p$.



Catchme:

Use Blue Path.

Blue with prob. 1/2.

Blue with prob. 1/3.

Green with prob. 1/2.

Green with prob. 1/6.

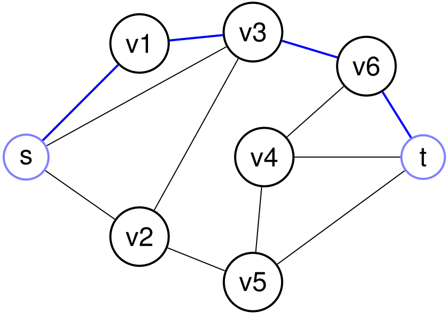
Pink with prob. 1/2.

Catcher:

Caught!

Caught, sometimes.

With probability 1/2.



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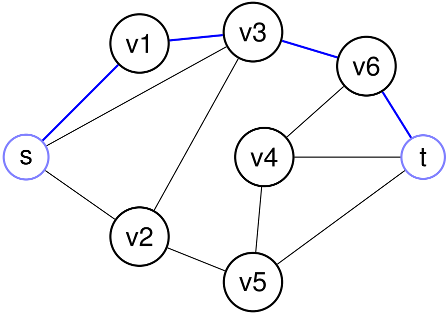

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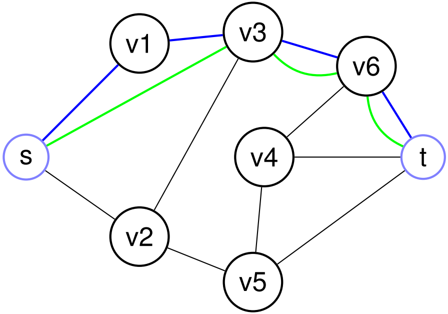

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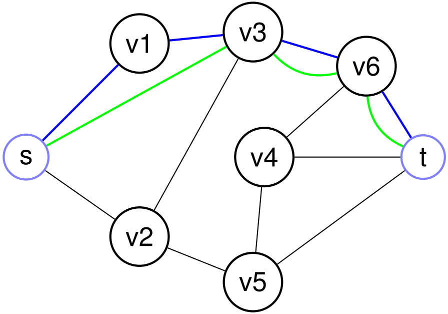

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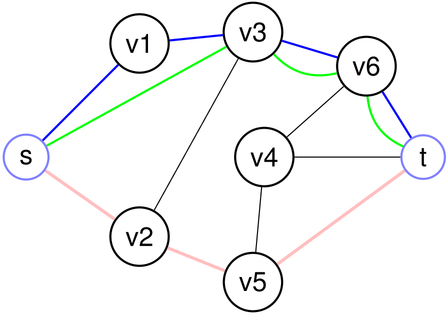

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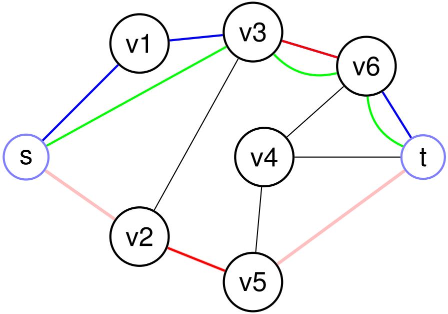

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Row solution: $Pr[p_1] = 1/2$, $Pr[p_2] = 1/3$, $Pr[p_3] = 1/6$.

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Offense

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Offense (Best Response.):

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Catch me: route along shortest path.
(Knows catcher's distribution.)

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Offense (Best Response.):

Catch me: route along shortest path.
(Knows catcher's distribution.)
Catcher: raise toll on most congested edge.

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Defense:

Where should "catcher" play to catch any path?

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Where should "catcher" play to catch any path? a cut.

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Where should "catcher" play to catch any path? a cut.
Minimum cut allows the maximum toll on any edge!

Example.

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What should "catch me" do to avoid catcher?

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Max-Flow Problem.

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Minimum cut allows the maximum toll on any edge!

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minimize maximum load on any edge!
Max-Flow Problem.

Note: exponentially many strategies for "catch me"!

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

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$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.

Toll/Congestion

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$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Toll/Congestion

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Row pays if column chooses edge on any path.

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$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.

Toll/Congestion

Given: $G = (V, E)$.
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Matrix:
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$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

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Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Again: exponentially (squared) number of paths for route player.

Summary...

You should now know about

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Games

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You should now know about

Games
Nash Equilibrium

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Finding Equilibrium.

...see you Tuesday.