Admin:

Check Piazza. There is a poll on bspace.

Today:
- Finish Path Routing.
- Games
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Today:
  - Finish Path Routing.
  - Games
Path Routing.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of $k$ paths connecting $s_i$ and $t_i$ and minimize max load on any edge.
Path Routing.

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Value: 3
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Path Routing.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of $k$ paths connecting $s_i$ and $t_i$ and minimize max load on any edge.
Toll problem.

Given $G = (V, E)$, $(s_1, t_1), \ldots, (s_k, t_k)$, find a set of $k$ paths assign one unit of “toll” to edges to maximize total toll for connecting pairs.
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Assign $\frac{1}{11}$ on each of 11 edges.
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Total toll:
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Can we do better?
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Assign $1/2$ on these two edges.
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Assign 1/2 on these two edges. 
Total toll: $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$
Toll is lower bound on Path Routing.

From before:
Max bigger than minimum weighted average:
\[ \max_e c(e) \geq \sum_e c(e)d(e) \]
Total length is total congestion:
\[ \sum_e c(e)d(e) = \sum_i d(p_i) \]
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Each path, \( p_i \), in routing has length \( d(p_i) \geq d(s_i, t_i) \).
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A toll solution is lower bound on any routing solution.
Toll is lower bound on Path Routing.

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Max bigger than minimum weighted average:
$$\max_e c(e) \geq \sum_e c(e)d(e)$$
Total length is total congestion: $$\sum_e c(e)d(e) = \sum_i d(p_i)$$
Each path, $$p_i$$, in routing has length $$d(p_i) \geq d(s_i, t_i)$$.  

$$\max_e c(e) \geq \sum_e c(e)d(e) = \sum_i d(p_i) \geq \sum_i d(s_i, t_i).$$

A toll solution is lower bound on any routing solution.  
Any routing solution is an upper bound on a toll solution.
Algorithm.

Assign tolls.

Assign tolls.

How to route?
Shortest paths!

Assign routing.

How to assign tolls?
Higher tolls on congested edges.

Toll: $d(e) \propto c(e)^2$.

Equilibrium: The shortest path routing has $d(e) \propto c(e)^2$.

The routing does not change, the tolls do not change.
Assign tolls.
How to route?

Equilibrium: The shortest path routing has
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d(e) \propto c(e).

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How to route? **Shortest paths!**
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How to assign tolls?
Algorithm.

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Toll: $d(e) \propto 2^{c(e)}$. 
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How to route? **Shortest paths!**
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**Equilibrium:**
Algorithm.

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The shortest path routing has \( d(e) \propto 2^{c(e)} \).
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How to route? **Shortest paths!**
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**Equilibrium:**
The shortest path routing has \( d(e) \propto 2^{c(e)} \).
The routing does not change, the tolls do not change.
How good is equilibrium?

Path is routed along shortest path and $d(e) \propto 2^{c(e)}$. 
How good is equilibrium?

Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.

$$c_{opt} \geq \sum_i d(s_i, t_i) = \sum_e d(e)c(e)$$
How good is equilibrium?

Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.

$$c_{opt} \geq \sum_i d(s_i, t_i) = \sum_e d(e) c(e)$$

$$= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e)$$
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Or $c_{max} \leq \left(1 + \frac{1}{m}\right)c_{opt} + 2\log m$.

(A)lmost within $2\log m$ of optimal!
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\[
\begin{align*}
\text{Let } c_t &= c_{\text{max}} - 2 \log m. \\
\text{Then, we have } c_{\text{opt}} &\geq \sum_i d(s_i, t_i) = \sum_e d(e) c(e) \\
&= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e) = \frac{\sum_e 2^{c(e)} c(e)}{\sum_e 2^{c(e)}} \\
&\geq \frac{\sum_{e: c(e) > c_t} 2^{c(e)} c(e)}{\sum_{e: c(e) > c_t} 2^{c(e)} + \sum_{e: c(e) \leq c_t} 2^{c(e)}}
\end{align*}
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How good is equilibrium?

Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.

For $e$ with $c(e) \leq c_{\text{max}} - 2 \log m; 2^{c(e)} \leq 2^{c_{\text{max}}-2\log m} = \frac{2^{c_{\text{max}}}}{m^2}$.

$$c_{\text{opt}} \geq \sum_i d(s_i, t_i) = \sum_e d(e)c(e)$$

$$= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e) = \frac{\sum_e 2^{c(e)} c(e)}{\sum_e 2^{c(e)}}$$

Let $c_t = c_{\text{max}} - 2 \log m$.

$$\geq \frac{\sum_{e: c(e) > c_t} 2^{c(e)} c(e)}{\sum_{e: c(e) > c_t} 2^{c(e)} + \sum_{e: c(e) \leq c_t} 2^{c(e)}}$$
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Path is routed along shortest path and $d(e) \propto 2^{c(e)}$.
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$$c_{opt} \geq \sum_i d(s_i, t_i) = \sum_e d(e) c(e)$$

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Let $c_t = c_{max} - 2 \log m$.

$$\geq \frac{\sum_{e : c(e) > c_t} 2^{c(e)} c(e)}{\sum_{e : c(e) > c_t} 2^{c(e)} + \sum_{e : c(e) \leq c_t} 2^{c(e)}}$$

$$\geq \frac{(c_t) \sum_{e : c(e) > c_t} 2^{c(e)}}{(1 + \frac{1}{m}) \sum_{e : c(e) > c_t} 2^{c(e)}}$$
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Path is routed along shortest path and \( d(e) \propto 2^{c(e)} \).
For \( e \) with \( c(e) \leq c_{\text{max}} - 2 \log m \); \( 2^{c(e)} \leq 2^{c_{\text{max}} - 2 \log m} = \frac{2^{c_{\text{max}}}}{m^2} \).

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\begin{align*}
c_{\text{opt}} & \geq \sum_i d(s_i, t_i) = \sum_e d(e) c(e) \\
& = \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e) = \frac{\sum_e 2^{c(e)} c(e)}{\sum_e 2^{c(e)}} \\
& \geq \frac{\sum_{e: c(e) > c_t} 2^{c(e)} c(e)}{\sum_{e: c(e) > c_t} 2^{c(e)} + \sum_{e: c(e) \leq c_t} 2^{c(e)}} \\
& \geq \frac{(c_t) \sum_{e: c(e) > c_t} 2^{c(e)}}{(1 + \frac{1}{m}) \sum_{e: c(e) > c_t} 2^{c(e)}} \\
& \geq \frac{(c_t)}{1 + \frac{1}{m}} = \frac{c_{\text{max}} - 2 \log m}{(1 + \frac{1}{m})} \\
\end{align*}
\]

Let \( c_t = c_{\text{max}} - 2 \log m \).
How good is equilibrium?

Path is routed along shortest path and \( d(e) \propto 2^{c(e)} \).

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Let \( c_t = c_{\text{max}} - 2 \log m \).

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\geq \frac{\sum_{e: c(e) > c_t} 2^{c(e)} c(e)}{\sum_{e: c(e) > c_t} 2^{c(e)} + \sum_{e: c(e) \leq c_t} 2^{c(e)}}
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\geq \frac{(c_t) \sum_{e: c(e) > c_t} 2^{c(e)}}{(1 + \frac{1}{m}) \sum_{e: c(e) > c_t} 2^{c(e)}}
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\geq \frac{(c_t) \sum_{e: c(e) > c_t} 2^{c(e)}}{1 + \frac{1}{m}} = \frac{c_{\text{max}} - 2 \log m}{(1 + \frac{1}{m})}
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Or \( c_{\text{max}} \leq (1 + \frac{1}{m}) c_{\text{opt}} + 2 \log m \).
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$c_{\text{opt}} \geq \sum_i d(s_i, t_i) = \sum_e d(e) c(e)

= \sum_e \frac{2^{c(e)}}{\sum_{e'} 2^{c(e')}} c(e) = \frac{\sum_e 2^{c(e)} c(e)}{\sum_e 2^{c(e)}}$

Let $c_t = c_{\text{max}} - 2 \log m$.

$\geq \frac{\sum_{e: c(e) > c_t} 2^{c(e)} c(e)}{\sum_{e: c(e) > c_t} 2^{c(e)} + \sum_{e: c(e) \leq c_t} 2^{c(e)}}$

$\geq \frac{(c_t) \sum_{e: c(e) > c_t} 2^{c(e)}}{(1 + \frac{1}{m}) \sum_{e: c(e) > c_t} 2^{c(e)}}$

$\geq \frac{(c_t)}{1 + \frac{1}{m}} = \frac{c_{\text{max}} - 2 \log m}{(1 + \frac{1}{m})}$

Or $c_{\text{max}} \leq (1 + \frac{1}{m}) c_{\text{opt}} + 2 \log m$.
(A)lmost) within $2 \log m$ of optimal!
The end: sort of.

Got to here in class. Feel free to continue reading.
Getting to equilibrium.

Maybe no equilibrium!
Getting to equilibrium.

Maybe no equilibrium!

Approximate equilibrium:
Getting to equilibrium.

Maybe no equilibrium!

**Approximate equilibrium:**

Each path is routed along a path with length within a factor of 3 of the shortest path and $d(e) \propto 2^{c(e)}$. 

We obtain $c_{\text{max}} = 3 \left(1 + \frac{1}{m}\right) c_{\text{opt}} + 2 \log m$. This is worse!

What do we gain?
Getting to equilibrium.

Maybe no equilibrium!

**Approximate equilibrium:**

Each path is routed along a path with length within a factor of 3 of the shortest path and $d(e) \propto 2^{c(e)}$.

Lose a factor of three at the beginning.
Getting to equilibrium.

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Each path is routed along a path with length within a factor of 3 of the shortest path and \( d(e) \propto 2^{c(e)} \).

Lose a factor of three at the beginning.

\[ c_{opt} \geq \sum_i d(s_i, t_i) \geq \frac{1}{3} \sum_e d(p_i). \]
Getting to equilibrium.

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What do we gain?
An algorithm!

Algorithm: reroute paths that are off by a factor of three.
(Note: $d(e)$ recomputed every rerouting.)
An algorithm!

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$p: w(p) = X$

$p': w(p') \leq X/3$
An algorithm!

Algorithm: reroute paths that are off by a factor of three. (Note: $d(e)$ recomputed every rerouting.)

Potential function: $\sum_e w(e)$, $w(e) = 2^{c(e)}$
An algorithm!

Algorithm: reroute paths that are off by a factor of three. (Note: \(d(e)\) recomputed every rerouting.)

\[ p: w(p) = X \]

Potential function: \( \sum_e w(e) \), \( w(e) = 2^{c(e)} \)

Moving path:
An algorithm!

Algorithm: reroute paths that are off by a factor of three. (Note: $d(e)$ recomputed every rerouting.)

$p$: $w(p) = X$ \[\implies w'(p) = X/2\]

$p'$: $w(p') \leq X/3$

Potential function: $\sum_e w(e)$, $w(e) = 2^{c(e)}$

Moving path:
Divides $w(e)$ along long path (with $w(p)$ of $X$) by two.
An algorithm!

Algorithm: reroute paths that are off by a factor of three.
(Note: $d(e)$ recomputed every rerouting.)

$p$: $w(p) = X$  \[\Rightarrow w'(p) = X/2\]

$-1$ for $c(e)$

$+1$ for $c(e)$

$p'$: $w(p') \leq X/3$  \[\Rightarrow w'(p') \leq 2X/3\]

Potential function: $\sum_e w(e)$, $w(e) = 2^{c(e)}$

Moving path:
Divides $w(e)$ along long path (with $w(p)$ of $X$) by two.
Multiplies $w(e)$ along shorter ($w(p) \leq X/3$) path by two.
An algorithm!

Algorithm: reroute paths that are off by a factor of three. (Note: \(d(e)\) recomputed every rerouting.)

-1 for \(c(e)\)

\[ p: w(p) = X \implies w'(p) = X/2 \]

+1 for \(c(e)\)

\[ p': w(p') \leq X/3 \implies w'(p') \leq 2X/3 \]

Potential function: \(\sum_e w(e), w(e) = 2^{c(e)}\)

Moving path:
Divides \(w(e)\) along long path (with \(w(p)\) of \(X\)) by two.
Multiplies \(w(e)\) along shorter \((w(p) \leq X/3)\) path by two.

\[ -\frac{X}{2} + \frac{X}{3} = -\frac{X}{6}. \]
An algorithm!

Algorithm: reroute paths that are off by a factor of three. (Note: \(d(e)\) recomputed every rerouting.)

\[-1 \text{ for } c(e)\]

\[p: w(p) = X \implies w'(p) = X/2\]

\[+1 \text{ for } c(e)\]

\[p': w(p') \leq X/3 \implies w'(p') \leq 2X/3\]

Potential function: \(\sum_e w(e), w(e) = 2^{c(e)}\)

Moving path:
Divides \(w(e)\) along long path (with \(w(p)\) of \(X\)) by two.
Multiplies \(w(e)\) along shorter (\(w(p) \leq X/3\)) path by two.

\[-\frac{X}{2} + \frac{X}{3} = -\frac{X}{6}\]

Potential function decreases.
An algorithm!

Algorithm: reroute paths that are off by a factor of three. (Note: $d(e)$ recomputed every rerouting.)

$-1$ for $c(e)$

$p$: $w(p) = X$

$\Rightarrow w'(p) = \frac{X}{2}$

$+1$ for $c(e)$

$p'$: $w(p') \leq \frac{X}{3}$

$\Rightarrow w'(p') \leq \frac{2X}{3}$

Potential function: $\sum_e w(e)$, $w(e) = 2^{c(e)}$

Moving path:
Divides $w(e)$ along long path (with $w(p)$ of $X$) by two.
Multiplies $w(e)$ along shorter ($w(p) \leq \frac{X}{3}$) path by two.

$-\frac{X}{2} + \frac{X}{3} = -\frac{X}{6}$.

Potential function decreases. $\implies$ termination and existence.
Replace $d(e) = (1 + \varepsilon)c(e)$.

Replace factor of 3 by $(1 + 2\varepsilon)c_{\text{max}} \leq (1 + 2\varepsilon)c_{\text{opt}} + 2 \log m / \varepsilon$. (Roughly)

Fractional paths?
Replace $d(e) = (1 + \varepsilon)^{c(e)}$. 
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Replace factor of 3 by $(1 + 2\varepsilon)$.
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Replace factor of 3 by $(1 + 2\varepsilon)$

$$c_{\text{max}} \leq (1 + 2\varepsilon)c_{\text{opt}} + 2\log m/\varepsilon. \quad \text{(Roughly)}$$
Replace $d(e) = (1 + \varepsilon)^{c(e)}$.

Replace factor of 3 by $(1 + 2\varepsilon)$

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Fractional paths?
Wrap up.

Dueling players:
Wrap up.

Dueling players:
Toll player raises tolls on congested edges.
Wrap up.

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Congestion player avoids tolls.
Wrap up.

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Converges to near optimal solution!
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A lower bound is “necessary” (natural),

Wrap up.
Wrap up.

Dueling players:
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Congestion player avoids tolls.

Converges to near optimal solution!

A lower bound is “necessary” (natural),
and helpful (mysterious?)!
Strategic Games.

\[ N \text{ players.} \]
Strategic Games.

$N$ players.
Each player has strategy set. $\{S_1, \ldots, S_N\}$. 

Example:
2 players
Player 1: \{Defect, Cooperate\).
Player 2: \{Defect, Cooperate\).
Payoff: \begin{align*} 
C & C(3,3) \\
D & C(0,5) \\
& D(5,0) \\
& C(1,1) 
\end{align*}
Strategic Games.

\(N\) players.
Each player has strategy set. \(\{S_1, \ldots, S_N\}\).
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Both cooperate. Payoff \((3,3)\).
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Both cooperate. Payoff (3, 3).

If player 1 wants to do better, what does he do?

Stable now! Nash Equilibrium: neither player has incentive to change strategy.
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E.G., OPEC, Airlines, .

More sophisticated models,
e.g., iterated dominance,
coalitions,
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Lots of interesting Game Theory!

This class (today): simpler version.
Digression..

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This class(today): simpler version.
Two Person Zero Sum Games

2 players.

Each player has strategy set:
- Player 1 has m strategies
- Player 2 has n strategies

Payoff function:
\[ u(i, j) = (-a, a) \] (or just \( a \)).

"Player 1 pays \( a \) to player 2."

Zero Sum: Payoff for any pair of strategies sums to 0.

Payoffs by \( m \) by \( n \) matrix: \( A \).

Row player minimizes, column player maximizes.

Roshambo: rock, paper, scissors.

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\end{array}
\]

Any Nash Equilibrium?

Two Person Zero Sum Games

2 players.

Each player has strategy set:
m strategies for player 1  n strategies for player 2

Payoff function:  $u(i, j) = (-a, a)$ (or just $a$).
“Player 1 pays $a$ to player 2.”

Zero Sum: Payoff for any pair of strategies sums to 0.

Payoffs by $m$ by $n$ matrix: $A$.
Row player minimizes, column player maximizes.

Roshambo: rock, paper, scissors.

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Any Nash Equilibrium?

Mixed Strategies.

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How do you play?
Mixed Strategies.

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How do you play?

Player 1: play each strategy with equal probability.
Mixed Strategies.

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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.
Mixed Strategies.

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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.
### Mixed Strategies.

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How do you play?

**Player 1:** play each strategy with equal probability.
**Player 2:** play each strategy with equal probability.

### Definitions.

**Mixed strategies:** Each player plays distribution over strategies.
Mixed Strategies.

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How do you play?

Player 1: play each strategy with equal probability.
Player 2: play each strategy with equal probability.

Definitions.

**Mixed strategies:** Each player plays distribution over strategies.

**Pure strategies:** Each player plays single strategy.
### Payoffs: Equilibrium.

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Payoffs?

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1. Remember zero sum games have one payoff.
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

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1 Remember zero sum games have one payoff.
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

Average Payoff.

\[ E[X] = \sum_{(i,j)} X(i,j) \Pr[(i,j)] \]

Each player chooses independently: \( \Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \).

\[ E[X] = 0 \]

\(^1\) Remember zero sum games have one payoff.
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

Average Payoff. Expected Payoff.

\[ \text{Expected Payoff} = \sum_{(i,j) \in \Omega} X(i,j) \Pr((i,j)) \]

Each player chooses independently: 
\[ \Pr((i,j)) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} \]

\[ E[X] = 0 \]

\(^1\) Remember zero sum games have one payoff.
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Payoffs? Can’t just look it up in matrix!.

Average Payoff. **Expected Payoff.**

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$

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\(^1\)Remember zero sum games have one payoff.
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

Average Payoff. **Expected Payoff.**

Sample space: \( \Omega = \{(i,j) : i,j \in [1,\ldots,3]\} \)

Random variable \( X \) (payoff).

\(^1\text{Remember zero sum games have one payoff.}\)
Payoffs: Equilibrium.

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Average Payoff. **Expected Payoff.**

Sample space: \( \Omega = \{(i,j) : i,j \in [1,\ldots,3]\} \)

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\[
E[X] = \sum_{(i,j)} X(i,j)\Pr[(i,j)].
\]

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\(^1\)Remember zero sum games have one payoff.
Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!. 

Average Payoff. **Expected Payoff.**

Sample space: $\Omega = \{(i,j) : i,j \in [1,..,3]\}$

Random variable $X$ (payoff).

$$E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)] .$$

Each player chooses independently:

---

¹Remember zero sum games have one payoff.
Payoffs: Equilibrium.

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Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

Average Payoff. Expected Payoff.

Sample space: \( \Omega = \{ (i, j) : i, j \in [1, \ldots, 3] \} \)

Random variable \( X \) (payoff).

\[
E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].
\]

Each player chooses independently:

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Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.
\]

\[
E[X] = 0.
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\(^1\)Remember zero sum games have one payoff.
### Payoffs: Equilibrium.

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Payoffs? Can’t just look it up in matrix!.

**Average Payoff. Expected Payoff.**

Sample space: \( \Omega = \{(i, j) : i, j \in [1, \ldots, 3]\} \)

Random variable \( X \) (payoff).

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E[X] = \sum_{(i,j)} X(i,j) Pr[(i,j)].
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Each player chooses independently:

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Pr[(i,j)] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.
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\[
E[X] = 0. \quad \text{\textsuperscript{1}}
\]

\textsuperscript{1}Remember zero sum games have one payoff.
Equilibrium

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Will Player 1 change strategy?
Will Player 1 change strategy? Mixed strategies uncountable!
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock?
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).
Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

Expected payoff of Paper?
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

Expected payoff of Paper? \( \frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0 \).
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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

Expected payoff of Paper? \( \frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0 \).

Expected payoff of Scissors?
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

Expected payoff of Paper? \( \frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0 \).

Expected payoff of Scissors? \( \frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0 \).
Equilibrium

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Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

Expected payoff of Paper? \( \frac{1}{3} \times -1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = 0 \).

Expected payoff of Scissors? \( \frac{1}{3} \times 1 + \frac{1}{3} \times -1 + \frac{1}{3} \times 0 = 0 \).

No better pure strategy.
Will Player 1 change strategy? Mixed strategies uncountable!

Expected payoffs for pure strategies for player 1.

Expected payoff of Rock? \( \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0 \).

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No better pure strategy. \( \implies \) No better mixed strategy!
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Mixed strat. payoff is weighted av. of payoffs of pure strats.
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Player 1 has no incentive to change! Same for player 2.
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Equilibrium!
Another example plus notation.

Rock, Paper, Scissors, prEmpt.

Payoffs:

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Equilibrium? (E,E).

Pure strategy equilibrium.

Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.
Another example plus notation.

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Another example plus notation.

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Equilibrium? \((E,E)\). Pure strategy equilibrium.

Notation:
Another example plus notation.

Rock, Paper, Scissors, prEempt.
PreEmpt ties preEmpt, beats everything else.

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Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Another example plus notation.

Rock, Paper, Scissors, prEempt.
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Notation: Rock is 1, Paper is 2, Scissors is 3, prEmpt is 4.
Payoff Matrix.

\[
A = \begin{bmatrix}
0 & 1 & -1 & 1 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & 1 \\
-1 & -1 & -1 & 0 \\
\end{bmatrix}
\]
Playing the boss...

Row has extra strategy: Cheat.
Playing the boss...

Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)
Playing the boss...

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\end{bmatrix}
\]

Note: column knows row cheats.
Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
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Rock is strategy 1, Paper is 2, Scissors is 3, and Cheat is 4 (for row.)

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Note: column knows row cheats.
Why play?
Row has extra strategy: Cheat. 
Ties with rock and scissors, beats paper. (Scissors, or no rock!)

Payoff matrix:
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Why play?
Row is column’s advisor.
Row has extra strategy: Cheat.
Ties with rock and scissors, beats paper. (Scissors, or no rock!)
Payoff matrix:
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Note: column knows row cheats.
Why play?
Row is column’s advisor.
... boss.
Playing the boss...

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Payoff matrix:
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Note: column knows row cheats.

Why play?
Row is column’s advisor.
... boss.
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \]

Equilibrium:

Row:
\((0, 1, 3), (1, 6), (1, 2)\).

Column:
\((1, 3), (1, 2), (1, 6)\).

Payoff?
Remember: weighted average of pure strategies.

Row Player.
Strategy 1:
\[ \frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times (-1) = \frac{1}{3} \]

Strategy 2:
\[ \frac{1}{3} \times (-1) + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6} \]

Strategy 3:
\[ \frac{1}{3} \times 1 + \frac{1}{2} \times (-1) + \frac{1}{6} \times 0 = -\frac{1}{6} \]

Strategy 4:
\[ \frac{1}{3} \times 0 + \frac{1}{2} \times 0 + \frac{1}{6} \times (-1) = -\frac{1}{6} \]

Payoff is 
\[ 0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6}) = -\frac{1}{6} \]

Column player: every column payoff is 
\(-\frac{1}{6}\).

Both only play optimal strategies!

Complementary slackness.

Why not play just one?
Change payoff for other guy!
Equilibrium: play the boss...

\[
A = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Equilibrium:
Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\).
Equilibrium: play the boss...

\[ A = \begin{bmatrix}
  0 & 1 & -1 \\
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\end{bmatrix} \]

Equilibrium:
Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).
Equilibrium: play the boss...

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Payoff?
Equilibrium: play the boss...

$$A = \begin{bmatrix}
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\end{bmatrix}$$

Equilibrium:
Row: $(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. Column: $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$.

Equilibrium: play the boss...

\[ A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

Equilibrium:
Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.
Equilibrium: play the boss...

$$A = \begin{bmatrix}
0 & 1 & -1 \\
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0 & 0 & -1
\end{bmatrix}$$

Equilibrium:
Row: $\left(0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$. Column: $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$.


Row Player.

Strategy 1: $\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1$
Equilibrium: play the boss...

\[ A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \]

Equilibrium:
Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


Row Player.
Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)
Equilibrium: play the boss...

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A = \begin{bmatrix}
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Equilibrium:
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Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)
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Row Player.

Strategy 1: \(\frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3}\)

Strategy 2: \(\frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6}\)
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Row Player.

Strategy 1: \( \frac{1}{3} \times 0 + \frac{1}{2} \times 1 + \frac{1}{6} \times -1 = \frac{1}{3} \)
Strategy 2: \( \frac{1}{3} \times -1 + \frac{1}{2} \times 0 + \frac{1}{6} \times 1 = -\frac{1}{6} \)
Strategy 3: \( \frac{1}{3} \times 1 + \frac{1}{2} \times -1 + \frac{1}{6} \times 0 \)
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Equilibrium:
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Row Player.

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Equilibrium:
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Row Player.

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Payoff is \(0 \times \frac{1}{3} + \frac{1}{3} \times (-\frac{1}{6}) + \frac{1}{6} \times (-\frac{1}{6}) + \frac{1}{2} \times (-\frac{1}{6})\)
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Column player: every column payoff is \(-\frac{1}{6}\).
Equilibrium: play the boss...

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Equilibrium:
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Both only play optimal strategies!
Equilibrium: play the boss...

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Row: \((0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2})\). Column: \((\frac{1}{3}, \frac{1}{2}, \frac{1}{6})\).


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Both only play optimal strategies! Complementary slackness.
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Column player: every column payoff is \(-\frac{1}{6}\).
Both only play optimal strategies! Complementary slackness.

Why not play just one?
Equilibrium: play the boss...

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Column player: every column payoff is \(-\frac{1}{6}\).

Both only play optimal strategies! Complementary slackness.

Why not play just one? Change payoff for other guy!
Lecture 2 ended here..and Lecture 3 reviewed a few of the previous slides and continued into lecture 3 notes.
Two person zero sum games.

$m \times n$ payoff matrix $A$. 

Row mixed strategy: $x = (x_1, \ldots, x_m)$.

Column mixed strategy: $y = (y_1, \ldots, y_n)$.

Payoff for strategy pair $(x, y)$:

$$p(x, y) = x^tAy.$$ 

That is,

$$\sum_i x_i \left( \sum_j a_{ij} y_j \right) = \sum_j \left( \sum_i x_i a_{ij} \right) y_j.$$ 

Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$$x^tAy^* = \max_y (x^tAy) = \min_x x^tAy^*.$$ 

(No better column strategy, no better row strategy.)
Two person zero sum games.

$m \times n$ payoff matrix $A$.

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Two person zero sum games.

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Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$(x^*)^t A y^* = \max_y (x^*)^t A y = \min_x x^t A y^* = \min_x (x^*)^t A y$.

(No better column strategy, no better row strategy.)
Two person zero sum games.

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**Two person zero sum games.**

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Recall row minimizes, column maximizes.  

Equilibrium pair: $(x^*, y^*)$?
Two person zero sum games.

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Payoff for strategy pair $(x, y)$:

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That is,

$$\sum_{i} x_i \left( \sum_{j} a_{i,j} y_j \right) = \sum_{j} \left( \sum_{i} x_i a_{i,j} \right) y_j.$$  

Recall row minimizes, column maximizes.

Equilibrium pair: $(x^*, y^*)$?

$$(x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*.$$  

(No better column strategy, no better row strategy.)
Equilibrium.

Equilibrium pair: $(x^*, y^*)$?

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p(x,y) = (x^*)^tAy^* = \max_y(x^*)^tAy = \min_x x^tAy^*.
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(No better column strategy, no better row strategy.)
Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^tAy^* = \max_y (x^*)^tAy = \min_x x^tAy^*. \]

(No better column strategy, no better row strategy.)

No row is better:
\[ \min_i A^{(i)} \cdot y = (x^*)^tAy^*. \]

\[ ^2 \text{A}^{(i)} \text{ is } i\text{th row.} \]
Equilibrium pair: \((x^*, y^*)\)?

\[ p(x, y) = (x^*)^t Ay^* = \max_y (x^*)^t Ay = \min_x x^t Ay^*. \]

(No better column strategy, no better row strategy.)

No row is better:
\[ \min_i A(i) \cdot y = (x^*)^t Ay^*. \]

No column is better:
\[ \max_j (A^t)(j) \cdot x = (x^*)^t Ay^*. \]

\(^2 A(i) \) is \( i \)th row.
Best Response

Column goes first:

\[ R = \max_y \min_x (x^tAy) \]

Note: \( x \) can be \((0,0,...,1,...0)\).

Example: Roshambo.

Value of \( R \)?

Row goes first:

\[ C = \min_x \max_y (x^tAy) \]

Again: \( y \) of form \((0,0,...,1,...0)\).

Example: Roshambo.

Value of \( C \)?
Best Response

**Column goes first:**
Find $y$, where best row is not too low..

$$R = \max_y \min_x (x^tAy).$$

---

**Note:** $x$ can be $(0, 0, \ldots, 1, \ldots, 0)$.

**Example:** Roshambo.

Value of $R$?

---

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^tAy).$$

---

Agin: $y$ of form $(0, 0, \ldots, 1, \ldots, 0)$.

**Example:** Roshambo.

Value of $C$?
Best Response

Column goes first:
Find $y$, where best row is not too low.

$$R = \max_{y} \min_{x} (x^t A y).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$. 

---

Example: Roshambo.

Value of $R$?

Row goes first:
Find $x$, where best column is not high.

$$C = \min_{x} \max_{y} (x^t A y).$$

Again: $y$ of form $(0, 0, \ldots, 1, \ldots 0)$.

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Example: Roshambo.

Value of $C$?
Best Response

Column goes first:
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Example: Roshambo.
Column goes first:
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Note: $x$ can be $(0,0,\ldots,1,\ldots 0)$.

Example: Roshambo. Value of $R$?
Best Response

Column goes first:
Find $y$, where best row is not too low..

\[ R = \max_y \min_x (x^t Ay). \]

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.
Example: Roshambo. Value of $R$?

Row goes first:
Find $x$, where best column is not high.
Best Response

**Column goes first:**
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.
Example: Roshambo. Value of $R$?

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$
Column goes first:
Find \( y \), where best row is not too low.

\[
R = \max_y \min_x (x^t Ay).
\]

Note: \( x \) can be \((0, 0, \ldots, 1, \ldots 0)\).

Example: Roshambo. Value of \( R \)?

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**Best Response**

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R = \max_y \min_x (x^t Ay).
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Example: Roshambo. Value of \( R \)?

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C = \min_x \max_y (x^t Ay).
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Example: Roshambo.
**Best Response**

**Column goes first:**
Find $y$, where best row is not too low.

$$R = \max_y \min_x (x^t Ay).$$

Note: $x$ can be $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshamo. Value of $R$?

**Row goes first:**
Find $x$, where best column is not high.

$$C = \min_x \max_y (x^t Ay).$$

Agin: $y$ of form $(0, 0, \ldots, 1, \ldots 0)$.

Example: Roshamo. Value of $C$?
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
Duality.

\[ R = \max_y \min_x (x^t A y). \]
\[ C = \min_x \max_y (x^t A y). \]

Weak Duality: \( R \leq C \).

Proof: Better to go second. At Equilibrium \((x^*, y^*)\), payoff \( v = \) row payoffs \((Ay^*) \) all \( \geq v = \Rightarrow R \geq v \).

column payoffs \((x^* \, t \, A) \) all \( \leq v = \Rightarrow v \geq C \).

\( \Rightarrow R \geq C \).

Equilibrium \( \Rightarrow R = C \).*

Strong Duality: There is an equilibrium point! and \( R = C \).*

Doesn't matter who plays first!
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
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**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
Duality.

\[ R = \max_y \min_x (x^t Ay) \]

\[ C = \min_x \max_y (x^t Ay) \]

**Weak Duality:** \( R \leq C \).

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At Equilibrium \((x^*, y^*)\), payoff \( v \): row payoffs \((Ay^*)\) all \(\geq v\)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \) \( \implies \) \( R \geq v \).

\( \square \)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \(v\):
- row payoffs \((Ay^*)\) all \(\geq v \implies R \geq v.\)
- column payoffs \(((x^*)^t A)\) all \(\leq v\)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

Weak Duality: \( R \leq C. \)

Proof: Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \) \( \implies \) \( R \geq v. \)
column payoffs \(((x^*)^t A)\) all \( \leq v \) \( \implies \) \( v \geq C. \)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
- Row payoffs \((Ay^*)\) all \(\geq v\) \(\implies R \geq v.\)
- Column payoffs \(((x^*)^t A)\) all \(\leq v\) \(\implies v \geq C.\)

\(\implies R \geq C\)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \(\geq v \implies R \geq v. \)
column payoffs \(((x^*)^t A)\) all \(\leq v \implies v \geq C. \)

\(\implies R \geq C\)

Equilibrium \(\implies R = C!\)
Duality.

\begin{align*}
R &= \max_y \min_x (x^t A y), \\
C &= \min_x \max_y (x^t A y).
\end{align*}

**Weak Duality:** $R \leq C$.

**Proof:** Better to go second.

At Equilibrium $(x^*, y^*)$, payoff $v$:

row payoffs $(A y^*)$ all $\geq v \implies R \geq v$.

column payoffs $((x^*)^t A)$ all $\leq v \implies v \geq C$.

$\implies R \geq C$

Equilibrium $\implies R = C$!

**Strong Duality:** There is an equilibrium point!
Duality.

\[ R = \max_y \min_x (x^t Ay). \]

\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \(v\):

row payoffs \((Ay^*)\) all \(\geq v\) \(\implies R \geq v.\)

column payoffs \(((x^*)^t A)\) all \(\leq v\) \(\implies v \geq C.\)

\(\implies R \geq C\)

Equilibrium \(\implies R = C!\)

**Strong Duality:** There is an equilibrium point! and \(R = C!\)
Duality.

\[ R = \max_y \min_x (x^t Ay). \]
\[ C = \min_x \max_y (x^t Ay). \]

**Weak Duality:** \( R \leq C. \)

**Proof:** Better to go second.

At Equilibrium \((x^*, y^*)\), payoff \( v \):
row payoffs \((Ay^*)\) all \( \geq v \) \(\implies\) \( R \geq v. \)
column payoffs \(((x^*)^t A)\) all \( \leq v \) \(\implies\) \( v \geq C. \)
\(\implies\) \( R \geq C \)

Equilibrium \(\implies\) \( R = C! \)

**Strong Duality:** There is an equilibrium point! and \( R = C! \)

Doesn’t matter who plays first!
Proof of Equilibrium.

Later. Let’s see some examples.
An “asymptotic” game.

“Catch me.”
An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.
Given $a, b \in V$.
Row (“Catch me”): choose path from $a$ to $b$.
Column (“Catcher”): choose edge.
Row pays if column chooses edge on path.
An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.
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Matrix:
row for each path: $p$
An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.
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Matrix:
row for each path: $p$
column for each edge: $e$
An “asymptotic” game.

“Catch me.”

Given: $G = (V, E)$.  
Given $a, b \in V$. 
Row ("Catch me"): choose path from $a$ to $b$.  
Column("Catcher"): choose edge. 
Row pays if column chooses edge on path.

Matrix:
row for each path: $p$
column for each edge: $e$
Catchme:
Use Blue Path
Blue with prob: 1/3.
Green with prob: 1/6.
Pink with prob: 1/2.

Catcher:
Caught! sometimes.
With probability 1/2.
Catchme:
Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:
Caught, sometimes.
With probability 1/2.
Catchme:
Blue with prob. $1/3$.
Green with prob. $1/6$.
Pink with prob. $1/2$.

Catcher:
Caught, sometimes.
With probability $1/2$. 
**Catchme:**

Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

**Catcher:**

Caught, sometimes.
With probability 1/2.
Catchme:
Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:
Caught, sometimes.
With probability 1/2.
**Catchme:**
Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

**Catcher:**
*Caught, sometimes.*
With probability 1/2.
Catchme:
Blue with prob. 1/3.
Green with prob. 1/6.
Pink with prob. 1/2.

Catcher:
Caught, sometimes.
   With probability 1/2.
Example.

Example.


Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$.
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)
Edge solution: \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)

Offense
Example.


Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

Offense (Best Response.):
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)
Edge solution: \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)

**Offense (Best Response.):**

Catch me: route along shortest path.
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)
Edge solution: \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)

**Offense (Best Response.):**

Catch me: route along shortest path.
   (Knows catcher’s distribution.)
Example.


Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.
   (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
Example.

Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.
   (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
   (Knows catch me’s distribution.)
Example.

Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

Offense (Best Response.):

Catch me: route along shortest path.
  (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
  (Knows catch me’s distribution.)

Defense:
Example.

Edge solution: $Pr[e_1] = 1/2, Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.   
   (Knows catcher’s distribution.)  
Catcher: raise toll on most congested edge.   
   (Knows catch me’s distribution.)

**Defense:**

Where should “catcher” play to catch any path?
Example.

Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.  
   (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.  
   (Knows catch me’s distribution.)

**Defense:**

Where should “catcher” play to catch any path? a cut.
Example.


Edge solution: $Pr[e_1] = 1/2$, $Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.
   (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
   (Knows catch me’s distribution.)

**Defense:**

Where should “catcher” play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!
Example.

Row solution: \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)
Edge solution: \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)

**Offense (Best Response.):**

Catch me: route along shortest path.
(Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
(Knows catch me’s distribution.)

**Defense:**

Where should “catcher” play to catch any path? a cut. **Minimum cut** allows the maximum toll on any edge!

What should “catch me” do to avoid catcher?
Example.


Edge solution: $Pr[e_1] = 1/2, Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.
  (Knows catcher’s distribution.)

Catcher: raise toll on most congested edge.
  (Knows catch me’s distribution.)

**Defense:**

Where should “catcher” play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!

What should “catch me” do to avoid catcher?
minimize maximum load on any edge!
Example.

Edge solution: $Pr[e_1] = 1/2, Pr[e_2] = 1/2$

**Offense (Best Response.):**

Catch me: route along shortest path.
   (Knows catcher’s distribution.)
Catcher: raise toll on most congested edge.
   (Knows catch me’s distribution.)

**Defense:**

Where should “catcher” play to catch any path? a cut.
**Minimum cut** allows the maximum toll on any edge!

What should “catch me” do to avoid catcher?
minimize maximum load on any edge!
**Max-Flow Problem.**
Example.

Row solution:  \( Pr[p_1] = 1/2, \ Pr[p_2] = 1/3, \ Pr[p_3] = 1/6. \)  
Edge solution:  \( Pr[e_1] = 1/2, \ Pr[e_2] = 1/2 \)  

**Offense (Best Response.):**  
Catch me: route along shortest path.  
   (Knows catcher’s distribution.)  
Catcher: raise toll on most congested edge.  
   (Knows catch me’s distribution.)  

**Defense:**  
Where should “catcher” play to catch any path? a cut.  
**Minimum cut** allows the maximum toll on any edge!  
What should “catch me” do to avoid catcher?  
minimize maximum load on any edge!  
**Max-Flow Problem.**  
Note: exponentially many strategies for “catch me”!
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

Offense: (Best Response.)
Toll/Congestion

Given: \( G = (V, E) \).
Given \((s_1, t_1) \ldots (s_k, t_k)\).
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: \( r \)
column for each edge: \( e \)

\( A[r, e] \) is congestion on edge \( e \) by routing \( r \)

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll/Congestion

Given: \( G = (V, E) \).
Given \((s_1, t_1) \ldots (s_k, t_k)\).
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: \( r \)
column for each edge: \( e \)

\( A[r, e] \) is congestion on edge \( e \) by routing \( r \)

**Offense:** *(Best Response.)*
Router: route along shortest paths.
Toll: charge most loaded edge.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**

**Router:** route along shortest paths.
**Toll:** charge most loaded edge.

**Defense:** Toll: maximize shortest path under tolls.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Router: route along shortest paths.
**Toll: charge most loaded edge.**

**Defense:** Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll: charge most loaded edge.

**Defense:** Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.
Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: $r$
column for each edge: $e$

$A[r, e]$ is congestion on edge $e$ by routing $r$

**Offense: (Best Response.)**
Router: route along shortest paths.
Toll: charge most loaded edge.

**Defense:** Toll: maximize shortest path under tolls.
Route: minimize max loaded on any edge.

Again: exponentially (squared) number of paths for route player.
Summary...

You should now know about...
Summary...

You should now know about
Games

Nash Equilibrium
Pure Strategies
Zero Sum Two Person Games
Mixed Strategies.
Checking Equilibrium.
Best Response.
Statement of Duality Theorem.
You should now know about

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Finding Equilibrium.

...see you Tuesday.